A STUDY OF GENERATING FUNCTION INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS OF ONE AND MORE VARIABLES

A THESIS

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by

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CERTIFICATE

I feel great pleasure in certifying that the thesis entitled "A STUDY OF GENERATING FUNCTION INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS OF ONE AND MORE VARIABLES", embodies a record of the results of investigations carried out by Mr. Deepak Kumar Kabra under my supervision. I am satisfied with the findings of present work.

He has completed the residential requirement as per rules.

I recommend the submission of thesis.

Date: August 2015

(Dr. Yasmeen)
Research Supervisor

DECLARATION

I hereby declare that the

(i) The thesis entitled *"A STUDY OF GENERATING FUNCTION INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS OF ONE AND MORE VARIABLES"* submitted by me is an original piece of research work, carried out under the supervision of Dr. Yasmeen.

(ii) The above thesis has not been submitted to this university or any other university for any degree.

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Contents

Chapter 1

A Brief Survey of the Literature

1.1 Introduction

The survey of literature is a major issue in taking up the research work in any discipline. For proposed work we have substantially ventured to go through the celebrated work of some eminent Mathematician around the globe for the past status of the work done on the subject the work available in literature.

Special functions and their applications are now awe-inspiring in their scope, variety and depth. Not only in their rapid growth in pure Mathematics and its applications to the traditional fields of Physics, Engineering and Statistics but in new fields of applications like Behavioral Science, Optimization, Biology, Environmental Science and Economics, etc. they are emerging.

The solution of many problems related to the various areas of Science (Theoretical and Experimental Physics), Mathematical Analysis, Applied Mathematics and Cybernetic Technology, etc. is reduced to the calculations of definite and indefinite integrals or to the summation of series containing elementary and special functions.

The Special function of Mathematical Physics appear often in solving partial differential equations by the method of separation of variables or in finding eigen functions of differential operators in certain curvilinear system of co-ordinates.

A special function is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated. Besides elementary functions such as x^n, e^x, $\ell n(x)$, and $\sin x$, higher functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) come under the category of special functions. In fact special functions are solutions of a wide class of mathematically and physically relevant functional equations. As far as the origin of special functions is concerned the special function of mathematical physics arises in the solution of partial differential equations governing the behavior of certain physical quantities.

Special functions have been used for several centuries, since they have numerous applications in astronomy, trigonometric functions which have been studied for over a thousand years. Even the series expansions for sine and cosine, as well as the arc tangent were known for long time ago from the fourteen century. Since then the subject of special functions has been continuously developed with contribution of several mathematicians including Euler, Legendre, Laplace, Gauss, Kummer, Riemann and Ramanujan. In the past several years the discoveries of new special functions and applications of this kind of functions to new areas of mathematics have initiated a great interest of this field.

In recent years, particular cases of long familiar special functions have been clearly defined and applied as orthogonal polynomials. The adjective special is used in this connection because here we are not, as in analysis, concerned with the general properties of functions, but only with the properties of functions which arise in the solution of special problems.

The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions.

The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries, the boundary - value problems of electromagnetism and heat in the nineteenth, and the eigenvalue problems of quantum mechanics in the twentieth.

Seventeenth-century England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of the gamma function long before Euler reached it. Euler found most of the major properties of the Gamma functions around 1730. In 1772 Euler evaluated the Beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss, respectively, early in the next century.

A major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gausss memoir on the $_2F_1$ series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The $_3F_2$ series was studied by Clausen (1928) and the $_1F_1$ series by Kummer (1836). The functions which Bessel considered in his memoir of 1824 are $_0F_1$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics, near the end of the century Appell (1880) introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.

This chapter attempts to give an introduction to the topic of study and brief survey of the contributions made by some of the earlier workers in this field. It shall not be our endeavour to give a complete chronological survey of all the developments in this field but shall mention only those relevant to the present work.

The present thesis is influenced by the work of Abramowitz-Stegun [1], Alassar [3], Alhaidari [4], Andrews [8], Andrews-Larry [9], Andrews L. C. [10], Appell-Kampé [12], Askey-Gasper [13], Bailey [17], Barr [19], Bateman [20-22], Bell [24], Brychkov [29], Burchnall-Chaundy [30], Carlitz [31], Carlson [36-38], Chandel [43], Chandel etal. [44], Chaudhary etal. [52], Chhabra-Rusia [57], Cohen [58,59], Cossar-Erdélyi [60], Delerue [76], Deshpande [77-78], Deshpande-Bhise [79], Dickinson-Warsi [82], Doetsch [84], Erdélyi [87, 88, 90], Erdélyi etal. [91-95], Exton [98, 105, 107, 109, 110], Fasenmyer-Celine [111,112], Feldheim [113], Fujiwara [115], Gauss [116], Gould [117], Gould-Hopper [118], Gradshteyn and Ryzhik [119], Gupta [120], Hái etal. [122], Hansen [123], Henrici [124], Horn [129-131], Humbert [132, 135], Jain [136], Kampé de Fériet [139], Karlsson [143, 144, 146], Karlsson etal. [151], Khan, I. A. [153-157], Khan-Shukla [165, 167, 168], Khan, M. S. [169], Khandekar [171], Kümmer [176], Lauricella [178], Lavoie etal. [180], Lebedev [182], Luke [184-185], Magnus etal. [188], Manocha, B. L. [189], Manocha, H. L. [190-192], Manocha-Sharma [193-195], Marichev [196], Mathai-Saxena [197], Mayr [198], Meijer [200-202], Miller [207], Nixon [210], Oberhettinger-Badii [211], Oldham-Spanier [212], Olver [213], Pasternack

[215], Pathan [216], Pathan etal. [219], Pathan-Yasmeen [223-224], Pearce-Potts [226], Prudnikov etal. [235-237], Qureshi-Ahmed [239], Qureshi etal. [243, 245, 249], Qureshi-Yasmeen [248], Ragab [250], Rainville [252], Rice[253], Roberts-Kaufman [254], Saran [257-261], Schl*ä*fli [262], Sharma, B.L. [264-266], Sharma etal. [267], Shrivastava, H. S. P. [271, 273, 274, 281, 283, 284, 286-288], Slater [293], Sneddon [294], Srivastava, H. M. [296-299, 301-304, 309, 312, 315], Srivastava-Daoust [319, 320], Srivastava-Exton [322], Srivastava-Karlsson [324], Srivastava-Manocha [326], Srivastava-Panda [328], Srivastava-Pathan [329, 330], Srivastava-Singhal [332], Szeg*ö* [333], Thakare [334], Thompson [335], Watson [336], Whittaker [337], Widder [338].

1.2 Gamma Function, Pochhammer Symbols and Associated Results

Throughout the present work, we use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$.
Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Gamma Function:

The Gamma function, introduced by Euler in 1720 when he solved the problem of extending the factorial function to all real or complex numbers, is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \Re(z) > 0 \tag{1.2.1}$$

Using (1.2.1), simple integration by parts, we get a recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z) \tag{1.2.2}$$

From (1.2.2), we get another useful result

$$\Gamma(n+1) = n! \qquad n = 0, 1, 2, 3, \ldots \tag{1.2.3}$$

Therefore Gamma function is the generalization of factorial function. The Gamma function has several equivalent definitions, most of which are due to Euler.

Euler's Infinite Product:

Euler's infinite product is defined as

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right\} \tag{1.2.4}$$

Weierstrass Infinite Product:

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\} \tag{1.2.5}$$

where γ =Euler-Mascheroni constant = 0.577215664901532860606512

$$\gamma = \lim_{n\longrightarrow\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right) \tag{1.2.6}$$

$$-\gamma = \Gamma'(1) = \int_0^{\infty} e^{-t}\ln(t)dt \tag{1.2.7}$$

Gauss's Limit Formula:

$$\Gamma(z) = \lim_{n\longrightarrow\infty}\frac{n!\, n^{z}}{z(z+1)(z+2)\ldots(z+n)}; \qquad z \neq 0,-1,-2,\ldots \tag{1.2.8}$$

If $z = 0,-1,-2,\ldots$ then $|\Gamma(z)| = \infty$ or equivalently $\frac{1}{|\Gamma(z)|} = 0$

Reflection Formula of Gamma Function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = -z\,\Gamma(z)\Gamma(-z) \tag{1.2.9}$$

where $z \neq 0, \pm1, \pm2, \ldots$.

Gauss Function:

$$\Pi(z) = \Gamma(z+1) \tag{1.2.10}$$

The Pochhammer Symbol:

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda,\nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \tag{1.2.11}$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Algebraic properties of Pochhammer's symbol

$$(a)_{p+q} = (a)_p(a+p)_q = (a)_q(a+q)_p \tag{1.2.12}$$

$$(b)_{-k} = \frac{(-1)^k}{(1-b)_k}, \quad (b \neq 0, \pm1, \pm2, \pm3, \ldots \quad and \quad k = 1, 2, 3, \ldots) \tag{1.2.13}$$

$$(n-k)! = \frac{(-1)^k (n)!}{(-n)_k}, \quad (0 \leq k \leq n) \tag{1.2.14}$$

Binomial Coefficient:

$$\binom{n}{r} = \frac{(n)!}{(r)!(n-r)!} = \frac{(-1)^r(-n)_r}{(r)!}, \qquad (0 \leq r \leq n) \tag{1.2.15}$$

Gauss's Multiplication Theorem for the Product of Gamma Functions: [252, p.26 Th. 10; 326, p.23 (27)]

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{(mz-\frac{1}{2})}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right) \tag{1.2.16}$$

where $m = 1, 2, 3, \ldots$ and $mz \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Gauss's Multiplication Theorem for the Product of Pochhammer Symbols: [326, p.23(26)]

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^{m} \left(\frac{\lambda+j-1}{m}\right)_n \tag{1.2.17}$$

where $n = 0, 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$

In particular, we have

$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n! \quad \text{and} \quad (2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n! \tag{1.2.18}$$

Legendre's Duplication Formula:

$$\sqrt{(\pi)}\,\Gamma(2z) = 2^{(2z-1)}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{1.2.19}$$

where $2z \in \mathbb{C}\backslash\mathbb{Z}_0^-$

Triplication Formula:

$$2\pi\ \Gamma(3z) = 3^{\left(3z-\frac{1}{2}\right)}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right) \tag{1.2.20}$$

where $3z \in \mathbb{C}\backslash\mathbb{Z}_0^-$

Psi Function (Digamma Function):

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.2.21}$$

$$\lim_{n\longrightarrow\infty} \frac{n^{(b-a)}\Gamma(a+n)}{\Gamma(b+n)} = 1 \tag{1.2.22}$$

Polygamma Functions:

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\psi(z) \tag{1.2.23}$$

Fractional Calculus:

Fractional calculus is the field of mathematical analysis, which deals with the investigation of integrals and derivatives of arbitrary order may be real or complex. The concept of fractional differintegral is an extension to the meaning of integration and differentiation from the integral order n of the operator $\frac{d^n}{dx^n}$ to an arbitrary order. In the initial stage of extension of meaning, the order n was taken to be a fraction and therefore this topic is also popularly known as fractional calculus. The well known monographs of Oldham and Spanier, Miller and Ross, Ross, Samko, Kilbas and Marichev and Nishimoto etc. provide a good account of fractional calculus.

The fractional integral of a function $f(x)$ is generally defined by the integral

$${}_{\alpha}D_x^{-\nu}\left[f(x)\right] = \frac{1}{\Gamma(\nu)}\int_{\alpha}^{x}(x-t)^{\nu-1}f(t)dt, \qquad \Re(\nu) > 0 \tag{1.2.24}$$

which is known as the Riemann-Liouville fractional integration of order ν.

The simplest approach to a definition of a fractional derivative commences with the formula

$$\frac{d^\alpha}{dx^\alpha}\{e^{\alpha x}\} = D_x^\alpha\{e^{\alpha x}\} = a^\alpha e^{\alpha x}, \tag{1.2.25}$$

where α is an arbitrary (real or complex) number.

The fractional derivative operator $D_z^{(b)}$ is an extension of the familiar derivative operator $D_z^{(n)}$ (n being a positive integer), to arbitrary (integer, rational, irrational and complex) values of b. The development of fractional derivative operators is receiving keen attention from many researchers presently. In particular, see for example, the work of Lavoie, et al. [180], Manocha [190], Manocha-Sharma [193, 194, 195], Oldham-Spanier [212], Sharma-Abiodun [267] and Deshpande [77]. In 1731, Euler extended the derivative formula in the following form.

$$D_z^n\{z^\lambda\} = \lambda(\lambda-1)\ldots(\lambda-n+1)\ z^{\lambda-n} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}\ z^{\lambda-n},$$
$$(n = 0, 1, 2, \ldots)$$

Let $D_z^{(b)}$ denotes the operator of fractional derivative having the arbitrary order b. The Riemann-Liouville fractional derivative of the power function holds that

$$D_z^{(b)}[z^{a-1}] = \frac{\Gamma(a)}{\Gamma(a-b)} z^{a-b-1}, \tag{1.2.26}$$
$$(a, a-b \in \mathbb{C}\backslash\mathbb{Z}_0^-)$$

The literature contains many examples of the use of fractional derivatives in the theory of hypergeometric functions, in solving ordinary and partial differential equations and integral equations.

1.3 Ordinary Generalized Hypergeometric Function of One Variable

A series which was beyond the ordinary geometric series

$$1 + z + z^2 + z^3 + \ldots \tag{1.3.1}$$

is called hypergeometric series. This term was first used by John Wallis in (1655), in his work " Arithmetica Infinitorum". In particular, he studied the series

$$1 + a + a(a+1) + a(a+1)(a+2) + \ldots \tag{1.3.2}$$

After J. Wallis, many other mathematicians studied the same series, notably the Swiss Mathematician L. Euler and others but it was C.F. Gauss, a famous German Mathematician who in (1812) studied the infinite series which is the generalization of the elementary geometric series and popularly known as Gauss hypergeometric series.

1.3.1 Power Series Form and Convergence Conditions

The generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters is defined by [326, p.42(1)]

$$ {}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!} $$

$$ = {}_pF_q\left(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2 \ldots, \beta_q; z\right) \tag{1.3.3} $$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2 \ldots, \beta_q$ take on complex values, provided that $\beta_j \neq 0, -1, -2, \ldots; \quad j = 1, 2, \ldots, q$.

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction, the ${}_pF_q$ series in (1.3.3)

(i) converges for $|z| < \infty$ if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$,

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Further, if we set

$$ \omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.3.4} $$

It is known that the ${}_pF_q$ series, with $p = q + 1$, is

(I) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,

(II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \Re(\omega) \leq 0$,

(III) divergent for $|z| = 1$, if $\Re(\omega) \leq -1$.

The Gauss hypergeometric function [326, p.29 (1.2.4)] is defined by the power series

$$ {}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (|z|<1; \quad a,b,c \neq 0,-1,-2,\ldots) \tag{1.3.5} $$

When $a=-p$ (p being a positive integer) then we get a hypergeometric polynomial in terminating form

$$ {}_2F_1[-p,b;c;z] = \sum_{n=0}^{p} \frac{(-p)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (b,c \neq 0,-1,-2,\ldots) $$

When $a=-p$ and $b=-q$ (p, q being positive integers) then we obtain a hypergeometric polynomial in another terminating form

$$ {}_2F_1[-p,-q;c;z] = \sum_{n=0}^{\min(p,q)} \frac{(-p)_n (-q)_n}{(c)_n} \frac{z^n}{n!}, \quad (c \neq 0,-1,-2,\ldots) $$

Gauss's Summation Theorem: [252, p.49 (Th. 18)]

$$ {}_2F_1\left[\begin{array}{c} a,b; \\ c; \end{array} 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.3.6} $$

where $Re(c-a-b)>0$ and $c \neq 0,-1,-2,\ldots$

Binomial Theorem:

In terms of hypergeometric function

$$ (1-t)^{-\alpha} = {}_1F_0\left[\begin{array}{c} \alpha; \\ -; \end{array} t\right] = \sum_{n=0}^{\infty} (\alpha)_n \frac{t^n}{n!} \quad , \qquad |t|<1 \tag{1.3.7} $$

An important special case when $p=q=1$, the above equation reduces to the confluent hypergeometric series ${}_1F_1$ named as Kummer's series given by

$$ {}_1F_1(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!}. \tag{1.3.8} $$

1.3.2 Mellin-Barnes type Contour Integral Representation for ${}_pF_q$

Mellin-Barnes integrals contain a group of products and quotients of gamma functions in the integrand. They were first introduced in 1888 by Pincherle

[227]. Their theory was developed and put on a firm footing through the works of Barnes[18] in 1907 and Mellin [204] in 1910. Dixon and Ferrar [83] in 1936 also made valuable contributions for further advancement of these integrals. A typical Mellin-Barnes integral according to celebrated monograph [91, p.49 (1)] is

$$f(z) = \frac{1}{2\,\pi\,\omega} \int_L \frac{\prod_{j=1}^{m} \Gamma(a_j + A_j\, s) \prod_{j=1}^{n} \Gamma(b_j - B_j\, s)}{\prod_{j=1}^{p} \Gamma(c_j + C_j\, s) \prod_{j=1}^{q} \Gamma(d_j - D_j\, s)} z^s\, ds \tag{1.3.9}$$

where details about the nature of contour L and other parameters can be referred in the monograph [91, p.49] and $\omega = \sqrt{(-1)}$.

The generalized hypergeometric function [252, p.100 Th. 35 and p.102 Th. 36] is defined by means of Mellin-Barnes type contour integral in the following form, when $p \leq q+1$ then

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}\,{}_pF_q\left[\begin{array}{c}\alpha_1, \alpha_2, \ldots, \alpha_p;\\ \beta_1, \beta_2, \ldots, \beta_q;\end{array} z\right]$$

$$= \frac{1}{2\pi\omega}\int_{L_1} \frac{(-z)^{\xi}\Gamma(-\xi)\Gamma(\alpha_1+\xi), \ldots, \Gamma(\alpha_p+\xi)}{\Gamma(\beta_1+\xi), \ldots, \Gamma(\beta_q+\xi)} d\xi \tag{1.3.10}$$

where L_1 is a suitable Mellin-Barnes path of integration [See 252, p. 95 figure (5), p. 98 figure(6)]and $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\{i = 1, 2, 3, \ldots, p,\ j = 1, 2, 3, \ldots, q\}$ and $\omega = \sqrt{-1}$, $z \neq 0$.
When $p = q+1$ then $|arg(-z)| < \pi$ and suppose that $|z| < 1$.
When $p = q$ then $|arg(-z)| < \frac{\pi}{2}$ i.e. $\Re(z) < 0$.
When $p < q$ then equation (1.3.10) is also valid.

1.4 Meijer's G-Function of One Variable

In 1946 through a series of papers, Meijer [203] extensively studied and developed a special case of the integral (1.3.9), by taking all the coefficient of s equal to 1 in the gamma functions occurring therein. This function is well known in the literature as Meijer G-function.

The G-function has acquired great popularly in the past sixty years. Its importance mainly lies in the fact that many of special functions useful in sciences and engineering notably, Bessel, Whittaker, Gauss hypergeometric, confluent hypergeometric, generalized hypergeometric functions, their combinations and other related functions, follow as special cases of the G-function. The reference [91, pp. 215-222] contains 75 such formulae expressing known functions by means of the G-symbol.

1.4.1 Mellin-Barnes type Contour Integral Representation for G-Function

In an attempt to give a meaning to the symbol ${}_pF_q$, when $p > q + 1$, Meijer introduced the G-function into Mathematical Analysis. Firstly the G-function was defined by Meijer [200] in the year 1936 by means of a finite series of generalized hypergeometric functions. Later on the Meijer's G-function of order (m, n, p, q) was defined by means of Mellin-Barnes type contour integral formula [91, p.207 (5.3.1); see also 184, p.143 (5.2.1); 197, p.2 (1.1.1, 1.1.3); 201, p.83; 202, p.1064 (21)], in the following form

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_n; a_{n+1}, \ldots, a_p \\ b_1, b_2, \ldots, b_m; b_{m+1}, \ldots, b_q \end{matrix}\right)$$

$$= \frac{1}{2\pi\omega} \int_{L_2} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds \tag{1.4.1}$$

$$(a_k - b_j \neq 1, 2, \ldots;\; k = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m)$$

where $0 \leq m \leq q, 0 \leq n \leq p\; ; z \neq 0$ and L_2 is a suitable contour (See three cases of contour in the monographs [91, p.207 (2,3,4); 184, p.144 (2,3,4); 235, p.617 (1,2,3,4)]) and an empty product is interpreted as 1 and the parameters are such that no pole of $\Gamma(b_j - s), \quad j = 1, 2, \ldots, m$ coincides with any pole of $\Gamma(1 - a_k + s), \quad k = 1, 2, \ldots, n$. Without any loss of generality, we are assuming that $p \leq q$.

There are three different paths L_2 of integration:

(1) L_2 runs from $-i\infty$ to $i\infty$ so that all poles of $\Gamma(b_j - s), \quad j = 1, 2, \ldots m$ lie to the right of the path and all the poles of $\Gamma(1 - a_k + s), \quad k = 1, 2, \ldots n$, lie to the left of the path L_2. The integral converges if $p+q < 2(m+n)$ and $|arg(z)| < \left(m + n - \frac{p}{2} - \frac{q}{2}\right)\pi$. If $|arg(z)| = \left(m + n - \frac{p}{2} - \frac{q}{2}\right)\pi \geq 0$ the integral converges absolutely when $p = q$, if $\Re[b_1 + b_2 + \ldots + b_q - a_1 - a_2 - \ldots - a_p] < -1$

(2) L_2 is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - s), \quad j = 1, 2, \ldots m$, once in the negative direction, but none of the poles of $\Gamma(1 - a_k + s), \quad k = 1, 2, \ldots n$. The integral converges if $q \geq 1$ and either $p < q$ or ($p = q$ and $|z| < 1$).

(3) L_2 is a loop starting and ending at $-\infty$ and encircling all poles of $\Gamma(1-a_k+s)$, $k = 1, 2, \ldots n$, once in the positive direction, but none of the poles $\Gamma(b_j-s), \quad j = 1, 2, \ldots m$. The integral converges if $p \geq 1$ and either $p > q$ or ($p = q$ and $|z| > 1$).

We shall always assume that the values of the parameters and of the variable x are such that at least one of the three conditions (1), (2), (3) makes sense. In cases when more than one of these conditions make sense, they lead to the same result so that there will be no ambiguity involved.

If in the definition of the G-function, the integrand only has factors with parameters (b_k) or only with (a_k), then the notation $G^{m,0}_{0,q}\left(z \,\middle|\, \begin{matrix} \cdot \\ (b_q) \end{matrix}\right)$ or $G^{0,n}_{p,0}\left(z \,\middle|\, \begin{matrix} (a_p) \\ \cdot \end{matrix}\right)$ is used.

The G-function is symmetric with respect to order of the parameters in four groups $a_1, a_2, \ldots, a_n \; ; \quad a_{n+1}, \ldots, a_p \; ; \; b_1, b_2, \ldots, b_m \; ; \quad b_{m+1}, \ldots, b_q$ individually.

If no pair among the parameters $b_1, b_2, \ldots, b_m$ may differ by an integer or zero (i.e. all poles are of the first order) then Meijer's function $G^{m,n}_{p,q}(z)$ can be expressed as a sum of m-generalized hypergeometric functions ${}_pF_{q-1}\left((-1)^{p-m-n}z\right)$ under the condition ($p < q$ and for all finite values of z) or ($p = q$ and $|z| < 1$).

If no pair among the parameters $a_1, a_2, \ldots, a_n$ may differ by an integer or zero (i.e. all poles are of the first order) then Meijer's function $G^{m,n}_{p,q}(z)$ can be expressed as a sum of n-generalized hypergeometric functions ${}_qF_{p-1}\left((-1)^{q-m-n}z^{-1}\right)$ under the condition ($p > q$ and for all finite values of z) or ($p = q$ and $|z| > 1$).

If one (or more) pair among the parameters $a_1, a_2, \ldots, a_n$ or $b_1, b_2, \ldots, b_m$ may differ by an integer or zero then Logarithmic forms of the G-function occur due to the appearance of the poles of the higher order than unity, in the integrand of contour integral (1.4.1).

If $p < q$ or $p = q, |z| < 1$ then $G^{0,n}_{p,q}\left(z \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right) = 0$, when $p > q$ or $p = q$ and $|z| > 1$ then $G^{m,0}_{p,q}\left(z \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right) = 0$ MacRobert's E-function never gained wide acceptance in the literature, mostly because it was found to be a special case of the Meijer's G-function.

$$E\left(p\,; a_1, a_2, \ldots, a_p : q; c_1, c_2, \ldots, c_q : z\right) = G^{p,1}_{q+1,p}\left(z \,\middle|\, \begin{matrix}1, c_1, c_2, \ldots, c_q\\ a_1, a_2, \ldots, a_p\end{matrix}\right) \tag{1.4.2}$$

1.4.2 Properties of G-Function

Property for cancellation of the numerator and denominator parameters

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_p\\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_{q-1}, a_1\end{matrix}\right) = G^{m,n-1}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix}a_2, a_3, \ldots, a_n, a_{n+1}, \ldots, a_p\\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_{q-1}\end{matrix}\right) \tag{1.4.3}$$

where $n, p, q \geq 1$.

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{p-1}, b_1\\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_q\end{matrix}\right) = G^{m-1,n}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix}a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{p-1}\\ b_2, b_3, \ldots, b_m, b_{m+1}, \ldots, b_q\end{matrix}\right) \tag{1.4.4}$$

where $m, p, q \geq 1$.

Translation property [202, p.1066 (24)]

$$z^{\sigma} G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_p\\ b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_q\end{matrix}\right)$$

$$= G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}a_1+\sigma, a_2+\sigma, \ldots, a_n+\sigma, a_{n+1}+\sigma, \ldots, a_p+\sigma\\ b_1+\sigma, b_2+\sigma, \ldots, b_m+\sigma, b_{m+1}+\sigma, \ldots, b_q+\sigma\end{matrix}\right) \tag{1.4.5}$$

Symmetric property (Transformation formula)

$$G_{p,q}^{m,n}\left(z\left|\begin{array}{c}a_1,a_2,\ldots,a_n,a_{n+1},\ldots,a_p\\ b_1,b_2,\ldots,b_m,b_{m+1},\ldots,b_q\end{array}\right.\right)$$

$$=G_{q,p}^{n,m}\left(\frac{1}{z}\left|\begin{array}{c}1-b_1,1-b_2,\ldots,1-b_m,1-b_{m+1},\ldots,1-b_q\\ 1-a_1,1-a_2,\ldots,1-a_n,1-a_{n+1},\ldots,1-a_p\end{array}\right.\right) \tag{1.4.6}$$

Properties of G-function [119, p.1034, (9.31.4)]

$$G_{p+1,q+1}^{m+1,n}\left(z\left|\begin{array}{c}a_1,a_2,\ldots,a_n,a_{n+1},\ldots,a_p,1-r\\ 0,b_1,b_2,\ldots,b_m,b_{m+1},\ldots,b_q\end{array}\right.\right)$$

$$=(-1)^r G_{p+1,q+1}^{m,n+1}\left(z\left|\begin{array}{c}1-r,a_1,a_2,\ldots,a_n,a_{n+1},\ldots,a_p\\ b_1,b_2,\ldots,b_m,b_{m+1},\ldots,b_q,1\end{array}\right.\right) \tag{1.4.7}$$

where $r=0,1,2,\ldots$.

Reduction formula between ${}_pF_q$ and G-functions [91, p.215 (5.6.1); 326, p.47 (9)], is given by when $p\leq q+1$, then

$$\frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_q)}{}_pF_q\left[\begin{array}{c}a_1,a_2,\ldots,a_p;\\ b_1,b_2,\ldots,b_q;\end{array}\ z\right]=G_{p,q+1}^{1,p}\left(-z\left|\begin{array}{c}1-a_1,1-a_2,\ldots,1-a_p\\ 0,1-b_1,1-b_2,\ldots,1-b_q\end{array}\right.\right)$$

$$=G_{q+1,p}^{p,1}\left(-\frac{1}{z}\left|\begin{array}{c}1,b_1,b_2,\ldots,b_q\\ a_1,a_2,\ldots,a_p\end{array}\right.\right)=(-z)^1G_{p,q+1}^{1,p}\left(-z\left|\begin{array}{c}-a_1,-a_2,\ldots,-a_p\\ -1,-b_1,-b_2,\ldots,-b_q\end{array}\right.\right) \tag{1.4.8}$$

where ($p\leq q$ and $|z|<\infty$) or ($p=q+1$ and $|z|<1$).
Thus every special function expressible in terms of the ${}_pF_q$ function is automatically contained in the G - function.

1.5 Fox-Wright Generalized Hypergeometric Function of One Variable

In 1933, E. M. Wright defined a more interesting generalized hypergeometric function of one variable [324, p.21(1.2.38, 1.2.40); 326, pp. 50-51(1.5.21), p.179

(34iii), p.395 (23)] and further generalizations of the series ${}_pF_q$ were given by Fox[114] and Wright [339, p.287; 340].

$$
{}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\Gamma(\alpha_2+A_2n)\ldots\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\Gamma(\beta_2+B_2n)\ldots\Gamma(\beta_q+B_qn)}\frac{z^n}{n!}
$$

$$
=H^{1,\ p}_{p,q+1}\left[-z\left|\begin{array}{ccc}(1-\alpha_1,A_1), & \ldots, & (1-\alpha_p,A_p)\\(0,1)(1-\beta_1,B_1),\ldots,(1-\beta_q,B_q)\end{array}\right.\right] \tag{1.5.1}
$$

where the coefficients $A_1, A_2, \ldots, A_p, B_1, B_2, \ldots, B_q$ are positive numbers and $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q$ are complex parameters. The function $H^{m,n}_{p,q}$ was given by C. Fox [114] in 1961 and Braaksma [27, p.278] gave the convergence condition.

$$
{}_p\Psi^*_q\left[\begin{array}{c}((\alpha_p,A_p));\\((\beta_q,B_q));\end{array} z\right]={}_p\Psi^*_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{array} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{nA_1}\ldots(\alpha_p)_{nA_p}}{(\beta_1)_{nB_1}\ldots(\beta_q)_{nB_q}}\frac{z^n}{n!} \tag{1.5.2}
$$

The Fox's H-function on the RHS of (1.5.1) makes sense when either

$$
\delta=(1+B_1+B_2+\ldots+B_q)-(A_1+A_2+\ldots+A_p)>0
$$

and

$$
0<|z|<\infty; \qquad z\neq 0
$$

The equality holds only for suitably constrained values of $|z|$ or appropriately bounded $|z|$ *i.e.* $\delta=0$ and $0<|z|<R={A_1}^{-A_1}{A_2}^{-A_2}\ldots{A_p}^{-A_p}{B_1}^{B_1}{B_2}^{B_2}\ldots{B_q}^{B_q}$

$$
{}_p\Psi_q\left[\begin{array}{c}(\alpha_1,1),\ldots,(\alpha_p,1);\\(\beta_1,1),\ldots,(\beta_q,1);\end{array} z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\\beta_1,\ldots,\beta_q;\end{array} z\right] \tag{1.5.3}
$$

1.6 Truncated Ordinary Hypergeometric Series

The truncated generalized Gaussian hypergeometric function of one variable is defined by:

$$
{}_mF_n\left[\begin{array}{c}(a_m);\\(b_n);\end{array} y\right] \text{ to } (i+1) \text{ terms} = {}_mF_n\left[\begin{array}{c}(a_m);\\(b_n);\end{array} y\right]_i=\sum_{k=0}^{i}\frac{[(a_m)]_k\ y^k}{[(b_n)]_k\ k!} \tag{1.6.1}
$$

where the suffix i indicates that only first $(i+1)$ terms of the F series are to be included in the expansion [34, p.437; 179, pp. 349-350; 181, p. 394(2); 205, p. 792; 206, p.430; 293, pp.83-84 (2.6.1.1, 2.6.1.7, 2.6.1.9)] and (h_r) represents the array of r parameters given by $h_1, h_2, \ldots, h_r$; $[(h_r)]_p$ means the product of r Pochhammer's symbols $(h_1)_p (h_2)_p \ldots (h_r)_p$.

1.7 Bilateral Hypergeometric Function

A further generalization of the hypergeometric series is provided in the form of the bilateral hypergeometric series (also called Dirichlet series or Laurent series). Though a few scattered result for what we now call the ordinary bilateral hypergeometric series were given by Dougall [86], yet a systematic study of such series was made first Bailey [16] in 1936.

The values of numerator and denominator parameters are adjusted in such away that each term of the following two F series [293, p.180 (6.1.1.2,6.1.1.3); see also 16; 3] is well-defined, then

$$\sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_A)_n \, z^n}{(b_1)_n (b_2)_n \ldots (b_B)_n} = \sum_{n=-\infty}^{\infty} \frac{[(a_A)_n] \, z^n}{[(b_B)_n]} = {}_A H_B \left[\begin{array}{c} (a_A); \\ (b_B); \end{array} z \right]$$

$$= {}_B H_A \left[\begin{array}{cc} 1-(b_B); & \frac{(-1)^{A-B}}{z} \\ 1-(a_A); & \end{array} \right]$$

$$= {}_{A+1}F_B \left[\begin{array}{c} (a_A), 1; \\ (b_B) \quad ; \end{array} z \right] + \frac{(-1)^{A-B} \prod_{i=1}^{B} (1-b_i)}{z \prod_{j=1}^{A} (1-a_j)} {}_{B+1}F_A \left[\begin{array}{cc} 2-(b_B), 1; & \frac{(-1)^{A-B}}{z} \\ 2-(a_A) \quad ; & \end{array} \right] \tag{1.7.1}$$

(i) The bilateral series ${}_A H_B$ is divergent, when $B \neq A$.

(ii) The bilateral series ${}_A H_B$ is convergent, when $B = A$ and $|z| = 1$. The function ${}_A H_A$ is defined for all real and complex values of numerator and denominator parameters except zero or integers and for all values of variable z such that $|z| = 1$.

(iii) The bilateral series ${}_AH_A$ is convergent, when

$\Re(b_1 + b_2 + \ldots + b_A - a_1 - a_2 - \ldots - a_A) > 1$ and $z = 1$.

(iv) The bilateral series ${}_AH_A$ is convergent, when

$0 < \Re(b_1 + b_2 + \ldots + b_A - a_1 - a_2 - \ldots - a_A) \leq 1$ and $z = -1$.

1.8 Appell's Double Hypergeometric Functions

The Appell's double hypergeometric function of first kind F_1 [105, p.23 (1.4.1); 326, p.53 (4)] is given by

$$F_1[a, b, c; d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.8.1}$$

$$F_1[a, b, c; d; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(d)_m} {}_2F_1\left[\begin{array}{c} a+m, c; \\ d+m; \end{array} y\right] \frac{x^m}{m!} \qquad max\{|x|, |y|\} < 1 \tag{1.8.2}$$

The Appell's double hypergeometric function of second kind F_2 [326, p.53 (5)]

$$F_2[a, b, c; d, g; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(g)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.8.3}$$

$$F_2[a, b, c; d, g; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(d)_m} {}_2F_1\left[\begin{array}{c} a+m, c; \\ g \ ; \end{array} y\right] \frac{x^m}{m!} \qquad \{|x| + |y|\} < 1 \tag{1.8.4}$$

The Appell's double hypergeometric function of third kind F_3 [326, p.53 (6)]

$$F_3[a, b, c, d; g; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n}{(g)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.8.5}$$

$$F_3[a, b, c, d; g; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(c)_m}{(g)_m} {}_2F_1\left[\begin{array}{c} b, \quad d; \\ g+m; \end{array} y\right] \frac{x^m}{m!} \qquad \max\{|x|, |y|\} < 1 \tag{1.8.6}$$

The Appell's function of fourth kind F_4 [326, p.53, (7)] is defined by

$$F_4[a, b; c, d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.8.7}$$

$$F_4[a, b; c, d; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1\left[\begin{array}{c} a+m, b+m; \\ d \qquad ; \end{array} y\right] \frac{x^m}{m!} \tag{1.8.8}$$

$$(\sqrt{|x|}+\sqrt{|y|}<1; \quad a,b,c,d \neq 0,-1,-2,\ldots)$$

When $a=-r$ (r being a positive intger) then we have Appell's polynomial of fourth kind

$$F_4[-r;b;c,d;x,y] = \sum_{m,n=0}^{m+n\leq r} \frac{(-r)_{m+n}(b)_{m+n}}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad (b,c,d\neq 0,-1,-2,\ldots) \tag{1.8.9}$$

1.9 Humbert's Double Hypergeometric Functions

Seven confluent forms of four Appell functions were defined in 1920 by P. Humbert [133]; see also [326, p.58 (36), (37), (40), (41), (42), (43), (44)]

$$\Phi_1[a,b;c;x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r}{(c)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.1}$$

where $|x|<1, \quad |y|<\infty$

$$\Phi_2[b,c;d;x,y] = \sum_{r,s=0}^{\infty} \frac{(b)_r(c)_s}{(d)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.2}$$

where $|x|<\infty, \quad |y|<\infty$

$$\Phi_3[b;d;x,y] = \sum_{r,s=0}^{\infty} \frac{(b)_r}{(d)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.3}$$

where $|x|<\infty, \quad |y|<\infty$.

$$\Psi_1[a,b;c,d;x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r}{(c)_r(d)_s}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.4}$$

where $|x|<1, \quad |y|<\infty$.

$$\Psi_2[a;c,d;x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}}{(c)_r(d)_s}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.5}$$

where $|x|<\infty, \quad |y|<\infty$.

$$\Xi_1[a,b,c;d;x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_r(b)_s(c)_r}{(d)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.6}$$

where $|x|<1, \quad |y|<\infty$.

$$\Xi_2[a,b;c;x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_r(b)_r}{(c)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.9.7}$$

where $|x|<1, \quad |y|<\infty$.

1.10 Kampé de Fériet's Double Hypergeometric Functions

Kampé de Fériet's double hypergeometric function of higher order in the modified notation of Srivastava and Panda [328, pp. 423(26), 424(27)], is given by

$$F^{p:q;k}_{j:m;n}\left[\begin{array}{c}(a_p):(b_q);(d_k);\\(g_j):(e_m);(h_n);\end{array}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{r+s}\prod_{i=1}^{q}(b_i)_r\prod_{i=1}^{k}(d_i)_s}{\prod_{i=1}^{j}(g_i)_{r+s}\prod_{i=1}^{m}(e_i)_r\prod_{i=1}^{n}(h_i)_s}\frac{x^r}{r!}\frac{y^s}{s!}\quad(1.10.1)$$

where (a_p) abbreviates the array of p parameters given by $a_1,a_2,\ldots,a_p$ with similar interpretations for $(b_q),(d_k)$ et cetera and for convergence of double hypergeometric series(1.10.1), we have

(i)$p+q<j+m+1$, $p+k<j+n+1$, $|x|<\infty$ and $|y|<\infty$ or

(ii)$p+q=j+m+1$, $p+k=j+n+1$, and

$$\begin{cases}|x|^{\frac{1}{p-j}}+|y|^{\frac{1}{p-j}}<1 & \text{if}\quad p>j\\ max\{|x|,|y|\}<1, & \text{if}\quad p\le j\end{cases}$$

The Appell's double hypergeometric functions F_1, F_2, F_3 and F_4 [91, p.224 (5.7.1.6,....,5.7.1.9)] are denoted by $F^{1:1;1}_{1:0;0}$, $F^{1:1;1}_{0:1;1}$, $F^{0:2;2}_{1:0;0}$ and $F^{2:0;0}_{0:1;1}$ respectively.

1.11 Exton's Double Hypergeometric Functions

In 1979, Exton [107, p.339(13)] defined the following double hypergeometric function

$$\mathscr{H}^{A:B;C;D}_{E:G;M;N}\left[\begin{array}{c}(a_A):(b_B);(c_C);(d_D);\\(e_E):(g_G);(m_M);(n_N);\end{array}x,y\right]$$

$$=\sum_{i,j=0}^{\infty}\frac{[(a_A)]_{2i+j}\,[(b_B)]_{i+j}\,[(c_C)]_i\,[(d_D)]_j}{[(e_E)]_{2i+j}\,[(g_G)]_{i+j}\,[(m_M)]_i\,[(n_N)]_j}\frac{x^i}{i!}\frac{y^j}{j!}\quad(1.11.1)$$

Making suitable adjustments in the numbers of numerator and denominator parameters of (1.11.1), we obtain Kampé de Fériet double hypergeometric function [328, p.423 (26); see also 329, p.23 (1.2,1.3); 330] given by $F^{B:C;D}_{G:M;N}=\mathscr{H}^{0:B;C;D}_{0:G;M;N}$, an additional double hypergeometric function of Exton [109, p.137 (1.2)] given by $X_{E:M;N}^{A:C;D} = \mathscr{H}_{E:0;M;N}^{A:0;C;D}$, Appell's four double hypergeometric functions [91, p.224 (6,7,8,9)] F_1, F_2, F_3, F_4. Humbert's double hypergeometric functions [91, pp.225-226 (20,21,22,23,24,25,26); see also 105; 324] $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Phi}_3, \mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{\Xi}_1, \mathbf{\Xi}_2$, Horn's non-confluent double hypergeometric functions [91, p.225 (15,16)] H_3 and H_4 and its confluent double hypergeometric functions [91, p.226 (34,35); see also 130] $\mathbf{H}_6$ and $\mathbf{H}_7$ respectively.

In 1982, Exton [109, p.137 (1.2)] defined the following double hypergeometric series

$$X_{E:G;H}^{A:B;D}\left[\begin{array}{ll}(a_A):(b_B);(d_D); & \\ & x,y\\ (e_E):(g_G);(h_H); & \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{2m+n}\,[(b_B)]_m\,[(d_D)]_n}{[(e_E)]_{2m+n}\,[(g_G)]_m\,[(h_H)]_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.11.2}$$

which is the generalization and unification of Horn's non-confluent double hypergeometric function[91, p.225 (16); see also 129; 130; 131] H_4 and Horn's confluent double hypergeometric function [91, p.226 (35); see also 131] $\mathbf{H}_7$.

In 1984, Exton [110, p.113 (1.2)] defined the double hypergeometric series of second order. The series is quite similar to one obtained by Kampé de Fériet in the year 1921 and is a generalization of Horn's non-confluent double hypergeometric functions G_2, H_2 [91, pp.224-225 (11,14)] and confluent hypergeometric functions $\mathbf{\Gamma}_1$,$\mathbf{\Gamma}_2$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_5$ and $\mathbf{H}_{11}$ [91, pp.226-227 (27,28,30,31,32,33,39)].

$$G_{E:G;H}^{A:B;D}\left[\begin{array}{ll}(a_A):(b_B);(d_D); & \\ & x,y\\ (e_E):(g_G);(h_H); & \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m-n}\,[(b_B)]_m\,[(d_D)]_n}{[(e_E)]_{m-n}\,[(g_G)]_m\,[(h_H)]_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.11.3}$$

where (a_A) and $[(a_A)]_{m-n}$ are defined in the same way as in the preceding section with similar interpretations for others.

1.12 Triple Hypergeometric Functions of Jain, Exton's and Srivastava

The triple hypergeometric function ${}_3\Phi_D^{(1)}$ of Jain [136, p.396 (2.9)] is the generalization of Humbert's double hypergeometric function Φ_1 and is defined by

$$ {}_3\Phi_D^{(1)}[a,b,c;d;x,y,z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_{r+s+k}(b)_r(c)_s}{(d)_{r+s+k}} \frac{x^r}{r!}\frac{y^s}{s!}\frac{z^k}{k!}. \tag{1.12.1} $$

Other notations of ${}_3\Phi_D^{(1)}$ are $\Phi_D^{(3)}$ of Srivastava and Exton [322, p.373 (12,13), see also 105, p.176 (5.10.9), 105, p.42 (2.1.1.3)] and F_{D_1} of Exton [98, p.81].

$$ \Phi_D^{(3)}[a,b,c,-;d;x,y,z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_{r+s+k}(b)_r(c)_s}{(d)_{r+s+k}} \frac{x^r}{r!}\frac{y^s}{s!}\frac{z^k}{k!} \tag{1.12.2} $$

$$ F_{D_1}[a,a,a;b,c,-;d,d,d;x,y,z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_{r+s+k}(b)_r(c)_s}{(d)_{r+s+k}} \frac{x^r}{r!}\frac{y^s}{s!}\frac{z^k}{k!}. \tag{1.12.3} $$

The triple hypergeometric function $\Phi_3^{(3)}$ of Exton [105, p.43 (2.1.1.5)] is the generalization of Humbert's double hypergeometric functions Φ_2 and Φ_3 and is defined by

$$ \Phi_3^{(3)}[a,b;c;x,y,z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_r(b)_s}{(c)_{r+s+k}} \frac{x^r}{r!}\frac{y^s}{s!}\frac{z^k}{k!}. \tag{1.12.4} $$

1.13 Srivastava's General Triple Hypergeometric Functions $F^{(3)}$

Triple hypergeometric function $F^{(3)}$ of Srivastava [301, p.428] is the unification and generalization of fourteen Lauricella's hypergeometric functions [178, pp.113-114, p.150] $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$, $F_D^{(3)}$ including Saran's ten triple hypergeometric functions [260, pp.293-294 (2.1, 2.2, ..., 2.10); see also 143; 257; 258; 259; 261] F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S, F_T, three additional functions H_A, H_B, H_C of Srivastava [302, pp.99-100; see also 296; 297; 298; 299], Confluent triple hypergeometric functions of Jain [136, p.396] and Exton [98, pp.80-81], extended definition of Saran's function F_K given by Sharma [264, p.613 (2)], Schläfli's function [262], Mayr's function [198, p.265], Manocha's function [189, p.86; see also 326, p.280 (29)] Φ_M, Erd$\acute{e}$lyi's function [326, p.222 (19); see also 88; 303, p.306 (2.3)] $\Phi^{(3)}$, Erd$\acute{e}$lyi's function [87, p.446 (7.2); see also 88] $\Phi_2^{(3)}$, Humbert's function [132, p.429; see also 332, p.356 (7)] $\Psi_2^{(3)}$, Gupta's function [120, p.169 (3.1); see also 326, p.240 (8); 312, p.401 (12)] Ψ_K, Exton's functions [105, p.43 (2.1.1.4,2.1.1.5)] $\Phi_3^{(3)}$, $\Xi_1^{(3)}$, Carlson's function of three variables [36, p.453 (2.1)] R, Srivastava-Exton function [322, p.373 (12,13)] $\Phi_D^{(3)}$, Appell's functions F_1, F_2, F_3, F_4, Humbert's confluent functions $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Phi}_3, \mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{\Xi}_1, \mathbf{\Xi}_2$, Kamp$\acute{e}$ de F$\acute{e}$riet's function [139; see also 12, p.150 (29)] $F^{A:B}_{C:D}$ in the notation of Burchnall and Chaundy [30, p.112], generalized Kamp$\acute{e}$ de F$\acute{e}$riet function $F^{A:B;G}_{C:D;H}$ in the slightly modified notation of Srivastava and Panda, Exton's function [105, p.89 (3.4.1,3.4.2)] ${}^{(K)}_{(1)}E^{(3)}_D$, ${}^{(K)}_{(2)}E^{(3)}_D$, Chandel's function [43, p.120 (2.3); see also 44] ${}^{(K)}_{(3)}E^{(3)}_C$, generalized hypergeometric function of one variable [293, p.41 (2.1.1.3)] ${}_AF_B$, Gauss ordinary hypergeometric function [116] ${}_2F_1$, K$\ddot{u}$mmer's confluent hypergeometric function [176] ${}_1F_1$, Bessel function [326, p.37 (9)] ${}_0F_1$, Exponential function [326, p.37 (10)] ${}_0F_0$ and Binomial function [326, p.44 (8)] ${}_1F_0$ etc. Triple series $F^{(3)}$ is given by:

$$F^{(3)}\left[\begin{array}{l}(a_A)::(b_B);(d_D);(e_E):(g_G);(h_H);(l_L);\\(m_M)::(n_N);(p_P);(q_Q):(r_R);(s_S);(t_T);\end{array}\ x,y,z\right]$$

$$=\sum_{i,j,k=0}^{\infty}\frac{[(a_A)]_{i+j+k}\,[(b_B)]_{i+j}\,[(d_D)]_{j+k}\,[(e_E)]_{k+i}\,[(g_G)]_i\,[(h_H)]_j\,[(l_L)]_k}{[(m_M)]_{i+j+k}\,[(n_N)]_{i+j}\,[(p_P)]_{j+k}\,[(q_Q)]_{k+i}\,[(r_R)]_i\,[(s_S)]_j\,[(t_T)]_k}\frac{x^i}{i!}\frac{y^j}{j!}\frac{z^k}{k!},\tag{1.13.1}$$

where (a_A) represents the array of A parameters given by $a_1a_2\dots a_A$ and $[(a_A)]_{i+j+k}$ represents the $(a_1)_{i+j+k}(a_2)_{i+j+k}\dots(a_A)_{i+j+k}$ with similar interpretations for others.

The triple hypergeometric series $F^{(3)}$ is convergent [169, pp.32-33; 57, p.156; 78, p.40; 326, p.70 (41)], when

$$\begin{cases}M+N+Q+R>A+B+E+G\\M+N+P+S>A+B+D+H\\M+P+Q+T>A+D+E+L\end{cases}$$

or equivalently

$$\begin{cases} 1+M+N+Q+R-A-B-E-G>0 \\ 1+M+N+P+S-A-B-D-H>0 \\ 1+M+P+Q+T-A-D-E-L>0 \end{cases}$$

where A, B, D, E, etc. are the non-negative integers and $|x| < 1$, $|y| < 1$ and $|z| < 1$; but in case

$$\begin{cases} 1+M+N+Q+R=A+B+E+G \\ 1+M+N+P+S=A+B+D+H \\ 1+M+P+Q+T=A+D+E+L \end{cases}$$

then $|x|$, $|y|$ and $|z|$ are to be restricted in such a way that the series involved, are either terminating or convergent.

1.14 Megumi Saigo's General Quadruple Hypergeometric Function $F_M^{(4)}$

In 1988, M. Saigo defined a more general quadruple hypergeometric function [256, p.15 (17); 255, pp.455-456 (16)] $F_M^{(4)}$ (slightly modified notation) in the following form:

$$F_M^{(4)}\left[\begin{matrix} (a_A):::(b_B)\;;(d_D)\;;(e_E)\;;\;(g_G)::(h_H)\;;\;(m_M)\;;\;(n_N)\;;\;(p_P);\;(q_Q)\;;\;(r_R): \\ (a'_{A'}):::(b'_{B'});(d'_{D'});(e'_{E'});(g'_{G'})::(h'_{H'});(m'_{M'});(n'_{N'});(p'_{P'});(q'_{Q'});(r'_{R'}): \end{matrix}\right.$$

$$\left.\begin{matrix} (s_S)\;;\;(t_T)\;;\;(u_U)\;;\;(w_W)\;; \\ (s'_{S'});(t'_{T'});(u'_{U'});(w'_{W'}); \end{matrix}\; x,\,y,\,z,\,c\right]$$

$$= \sum_{i,j,k,\ell=0}^{\infty} \frac{[(a_A)]_{i+j+k+\ell}[(b_B)]_{i+j+k}[(d_D)]_{j+k+\ell}[(e_E)]_{k+\ell+i}[(g_G)]_{\ell+i+j}[(h_H)]_{i+j}}{[(a'_{A'})]_{i+j+k+\ell}[(b'_{B'})]_{i+j+k}[(d'_{D'})]_{j+k+\ell}[(e'_{E'})]_{k+\ell+i}[(g'_{G'})]_{\ell+i+j}[(h'_{H'})]_{i+j}} \times$$

$$\times \frac{[(m_M)]_{i+k}[(n_N)]_{i+\ell}[(p_P)]_{j+k}[(q_Q)]_{j+\ell}[(r_R)]_{k+\ell}[(s_S)]_i[(t_T)]_j[(u_U)]_k[(w_W)]_\ell\; x^i\; y^j\; z^k\; c^\ell}{[(m'_{M'})]_{i+k}[(n'_{N'})]_{i+\ell}[(p'_{P'})]_{j+k}[(q'_{Q'})]_{j+\ell}[(r'_{R'})]_{k+\ell}[(s'_{S'})]_i[(t'_{T'})]_j[(u'_{U'})]_k[(w'_{W'})]_\ell\; i!\; j!\; k!\; \ell!} \tag{1.14.1}$$

It is the generalization and unification of Pathan's quadruple hypergeometric function $F_P^{(4)}$, Srivastava's quadruple hypergeometric function $F^{(4)}$, Quadruple hypergeometric functions of Bhati-Purohit [25,26], Chandel-Agarwal-Kumar [46] and Sharma-Parihar [268,269] etc.

1.15 Lauricella Hypergeometric Function of r Variables

One of the Lauricella's hypergeometric functions of r variables is denoted by $F_A^{(r)}$ and is defined by

$$F_A^{(r)}\left[a;b_1,\ldots,b_r;c_1,\ldots,c_r;z_1,\ldots,z_r\right]=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(a)_{k_1+\ldots+k_r}(b_1)_{k_1}\ldots(b_r)_{k_r}}{(c_1)_{k_1}\ldots(c_r)_{k_r}}\frac{z_1^{k_1}}{k_1!}\ldots\frac{z_r^{k_r}}{k_r!} \tag{1.15.1}$$

where

$$(|z_1|+\ldots+|z_r|<1),$$

In 1920-21 P. Humbert defined multi-variable hypergeometric function $\psi_2^{(r)}$ in the form [326, p.62 (11)]

$$\psi_2^{(r)}\left[a;c_1,c_2,\ldots,c_r;z_1,z_2,\ldots,z_r\right]$$

$$=\sum_{k_1,k_2,\ldots,k_r=0}^{\infty}\frac{(a)_{k_1+k_2+\ldots+k_r}}{(c_1)_{k_1}(c_2)_{k_2}\ldots(c_r)_{k_r}}\frac{z_1^{k_1}z_2^{k_2}\ldots z_r^{k_r}}{(k_1)!(k_2)!\ldots(k_r)!} \tag{1.15.2}$$

where $|z_1|<\infty,\quad |z_2|<\infty,\ldots,|z_r|<\infty.$

The denominator parameters are neither zero nor negative integers, the numerator parameters may be zero and negative integers.
Humbert's confluent hypergeometric function of r variables is denoted by $\Psi_2^{(r)}$ and is defined as [324, p.35 (1.4.11)]

$$\Psi_2^{(r)}\left[a;c_1,\ldots,c_r;x_1,\ldots,x_r\right]=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(a)_{k_1+\ldots+k_r}}{(c_1)_{k_1}\ldots(c_r)_{k_r}}\frac{x_1^{k_1}}{k_1!}\ldots\frac{x_r^{k_r}}{k_r!}$$

$$=\lim_{min(|b_1|,\ldots,|b_r|)\longrightarrow\infty}\left\{F_A^{(r)}\left[a;b_1,\ldots,b_r;c_1,\ldots,c_r;\frac{x_1}{b_1},\ldots,\frac{x_r}{b_r}\right]\right\} \tag{1.15.3}$$

1.16 Karlsson's Multivariable Hypergeometric Functions

Out of the many different hypergeometric functions of several variables the symmetric ones have attracted the greatest attention because symmetry may be expected to imply simple and elegant results. In the year 1893 four hypergeometric functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}, F_D^{(n)}$ introduced by Lauricella [178] are classical; other instances are the double hypergeometric function G_3 defined by Horn [130] and the triple hypergeometric functions H_B and H_C introduced by Srivastava [301]. The later admit themselves of natural generalizations $H_B^{(n)}$ and $H_C^{(n)}$.

The functions $H_B^{(n)}$ and $H_C^{(n)}$ are defined by the multiple series

$$H_B^{(n)}[a_1,\ldots,a_n;c_1,\ldots,c_n;x_1,\ldots,x_n] = \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{i=1}^{n} \frac{(a_i,k_i+k_{i-1})x_i^{k_i}}{(c_i,k_i)k_i!} \tag{1.16.1}$$

$$H_C^{(n)}[a_1,\ldots,a_n;c;x_1,\ldots,x_n] = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{(c,k_1+\ldots+k_n)} \prod_{i=1}^{n} \frac{(a_i,k_i+k_{i-1})x_i^{k_i}}{k_i!} \tag{1.16.2}$$

where $n \geq 2$ and

$$k_0 = k_n \tag{1.16.3}$$

In the sequel, cyclicity rules like (1.16.3) are understood.

The following particular cases for special values of n are readily established: (i) $H_B^{(2)}$ is Appell's F_4, (ii) $H_C^{(2)}$ is Gauss's ${}_2F_1$ with variable $x_1 + x_2$, (iii) $H_B^{(3)}$ and $H_C^{(3)}$ are the Srivastava functions H_B, H_C respectively and (iv) $H_C^{(4)}$ is the quadruple hypergeometric function K_{16} studied by Exton [101].

The function $H_C^{(n)}$ was introduced by the Karlsson, who obtained some, but not all, of its properties here given [145, 146]. The function $H_B^{(n)}$ was defined by Khichi [172]. The particular case $n = 2m$ (that is, the "even-dimensional" functions $H_B^{(2m)}$ and $H_C^{(2m)}$) was investigated by the Karlsson [147 (notation: F_{g_4} and F_b, respectively)]. Incidentally, it will be seen below that for $n = 2m$, certain additional results exist.

Moreover, the functions $H_B^{(n)}$ and $H_C^{(n)}$ are members of the extensive class of hypergeometric functions introduced by Srivastava and Daoust [319 (Section 4)].

Karlsson's quadruple hypergeometric function $H_c^{(4)}$ [146, p.37 (2.1)]

$$H_c^{(4)}\left[a_1, a_2, a_3, a_4; c\ ; x_1, x_2, x_3, x_4\right]$$

$$= \sum_{k_1,k_2,k_3,k_4=0}^{\infty} \frac{(a_1)_{k_4+k_1}(a_2)_{k_1+k_2}(a_3)_{k_2+k_3}(a_4)_{k_3+k_4}}{(c)_{k_1+k_2+k_3+k_4}} \frac{x_1^{k_1}}{(k_1)!}\frac{x_2^{k_2}}{(k_2)!}\frac{x_3^{k_3}}{(k_3)!}\frac{x_4^{k_4}}{(k_4)!} \quad (1.16.4)$$

1.17 Multi-Variable Extension of Kampé de Fériet's Functions

The multi-variable extension of Kampé de Fériet function is given in the form [319; see also 326, pp. 65-66]

$$F^{p:q_1;q_2;\ldots;q_n}_{\ell:m_1;m_2;\ldots;m_n}\left[\begin{array}{ll}(a_p) : (b^{(1)}_{q_1}); \ldots ; (b^{(n)}_{q_n}); & \\ & x_1, \ldots, x_n \\ (\alpha_\ell) : (\beta^{(1)}_{m_1}); \ldots ; (\beta^{(n)}_{m_n}); & \end{array}\right]$$

$$= \sum_{s_1,\ldots,s_n=0}^{\infty} \Lambda(s_1, \ldots, s_n) \frac{{x_1}^{s_1}}{s_1!} \cdots \frac{{x_n}^{s_n}}{s_n!} \quad (1.17.1)$$

$$\Lambda(s_1, \ldots, s_n) = \frac{\prod_{j=1}^{p}(a_j)_{s_1+\ldots+s_n} \prod_{j=1}^{q_1}(b_j^{(1)})_{s_1} \cdots \prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{\ell}(\alpha_j)_{s_1+\ldots+s_n} \prod_{j=1}^{m_1}(\beta_j^{(1)})_{s_1} \cdots \prod_{j=1}^{m_n}(\beta_j^{(n)})_{s_n}}$$

and, for convergence of the multiple hypergeometric series in (1.17.1),

$$1 + \ell + m_k - p - q_k \geqq 0, \qquad k = 1, \ldots, n;$$

the equality holds when, in addition, either

$$p > \ell \qquad \text{and} \qquad |x_1|^{\frac{1}{p-\ell}} + \ldots + |x_n|^{\frac{1}{p-\ell}} < 1$$

or

$$p \leqq \ell \qquad \text{and} \qquad max\{|x_1|, \ldots, |x_n|\} < 1$$

1.18 Srivastava-Daoust Multi-Variable Hypergeometric Functions

A further generalization of the Kampé de Fériet function is due to Srivastava and Daoust who indeed, defined an extension of Wright's ${}_p\psi_q$ function in two variables. More generally recalling here the following extension of the Wright's function ${}_p\psi_q$ in *several* variables, which is referred in the literature as the generalized Lauricella function of several variables, it is also due to Srivastava and Daoust [319, p.454].

$$F^{A\,:\,B^{(1)}\;;\ldots;\;B^{(n)}}_{C\,:\,D^{(1)}\;;\ldots;\;D^{(n)}}\left(\begin{array}{l}[(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\phi^{(1)}];\ldots;[(b^{(n)}):\phi^{(n)}];\\ \\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:[(d^{(1)}):\delta^{(1)}];\ldots;[(d^{(n)}):\delta^{(n)}];\end{array}\quad z_1,\ldots,z_n\right)$$

$$=\sum_{m_1,\ldots,\,m_n=0}^{\infty}\Omega(m_1,\ldots,\,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!} \tag{1.18.1}$$

where, for convenience,

$$\Omega(m_1,\ldots,\,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \tag{1.18.2}$$

the coefficients

$$\left\{\begin{array}{c}\theta_j^{(k)},j=1,\ldots,A;\phi_j^{(k)},j=1,\ldots,B^{(k)};\psi_j^{(k)},j=1,\ldots,C;\\ \delta_j^{(k)},j=1,\ldots,D^{(k)};\forall\;\;k\;\in\{1,\ldots,n\}\end{array}\right. \tag{1.18.3}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)},j=1,\ldots,B^{(k)};\forall\;\;k\;\in\{1,\ldots,n\}$$

with similar interpretations for (c) and $(d^{(k)}),k=1,\ldots,n$; *et cetera*. For the precise conditions under with the multiple hypergeometric series in (1.18.1) converges absolutely [see 320, pp. 157-158; see also 122].

1.19 Hypergeometric Forms of Some Elementary and Composite Functions

For hypergeometric forms of elementary and composite functions, we refer the monographs [1, 29, 91, 182, 184, 188, 235, 252, 293, 324, 326].

$$\sinh(t) = t\ {}_0F_1\left[\begin{matrix} -; & \frac{t^2}{4} \\ \frac{3}{2}\ ; & \end{matrix}\right] \tag{1.19.1}$$

$$\cosh(t) = {}_0F_1\left[\begin{matrix} -; & \frac{t^2}{4} \\ \frac{1}{2}\ ; & \end{matrix}\right] \tag{1.19.2}$$

$$Arth(t) = \tanh^{-1}(t) = \frac{1}{2}\ell_n\left(\frac{1+t}{1-t}\right) = t\ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & 1; & t^2 \\ \frac{3}{2} & ; & \end{matrix}\right] \tag{1.19.3}$$

$$Arsh(t) = \sinh^{-1}(t) = \ell_n\left(t + \sqrt{(1+t^2)}\right) = t\ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}; & -\ t^2 \\ \frac{3}{2} & ; & \end{matrix}\right] \tag{1.19.4}$$

$$\frac{\sin^{-1}(t)}{\sqrt{(1-t^2)}} = t\ {}_2F_1\left[\begin{matrix} 1, & 1; & t^2 \\ \frac{3}{2} & ; & \end{matrix}\right] \tag{1.19.5}$$

$$\frac{\sinh^{-1}(t)}{\sqrt{(1+t^2)}} = t\ {}_2F_1\left[\begin{matrix} 1, & 1; & -\ t^2 \\ \frac{3}{2} & ; & \end{matrix}\right] \tag{1.19.6}$$

$$\sin(t^m) = t^m\ {}_0F_1\left[\begin{matrix} -\ ; & \frac{-t^{2m}}{4} \\ \frac{3}{2}\ ; & \end{matrix}\right] \tag{1.19.7}$$

$$\cos(t^m) = {}_0F_1\left[\begin{matrix} -\ ; & \frac{-t^{2m}}{4} \\ \frac{1}{2}\ ; & \end{matrix}\right] \tag{1.19.8}$$

$$\exp\left(t^m\right) = e^{t^m} = {}_0F_0\left[\begin{matrix} -\ ; & t^m \\ -\ ; & \end{matrix}\right] \tag{1.19.9}$$

$$\arcsin(t) = \sin^{-1}(t) = t\ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}; \\ \frac{3}{2} & ; \end{matrix}\ t^2\right] \tag{1.19.10}$$

$$\arctan(t) = \tan^{-1}(t) = t\ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & 1; \\ \frac{3}{2} & ; \end{matrix}\ -t^2\right] = \sin^{-1}\left(\frac{t}{\sqrt{(1+t^2)}}\right) \tag{1.19.11}$$

$$-\frac{4}{t}\ell_n\left[\frac{1+\sqrt{(1-t)}}{2}\right] = {}_3F_2\left[\begin{matrix} 1, & 1, & \frac{3}{2}; \\ 2, & & 2\ ; \end{matrix}\ t\right] \tag{1.19.12}$$

$$\ell n(1+t) = t\ {}_2F_1\left[\begin{matrix} 1, & 1; \\ 2 & ; \end{matrix}\ -t\right] \qquad |t| < 1 \tag{1.19.13}$$

$$\ell n(1-t) = -t\ {}_2F_1\left[\begin{matrix} 1, & 1; \\ 2 & ; \end{matrix}\ t\right] \qquad |t| < 1 \tag{1.19.14}$$

$$\left[\sin^{-1}(t)\right]^2 = t^2\ {}_3F_2\left[\begin{matrix} 1, 1, 1; \\ \frac{3}{2}, \quad 2; \end{matrix}\ t^2\right] \tag{1.19.15}$$

$$\left[\sinh^{-1}(t)\right]^2 = t^2\ {}_3F_2\left[\begin{matrix} 1, 1, 1; \\ \frac{3}{2}, \quad 2; \end{matrix}\ -t^2\right] \tag{1.19.16}$$

$$\cos[m\{\sin^{-1}(t)\}] = {}_2F_1\left[\begin{matrix} \frac{-m}{2}, \frac{m}{2}; \\ \frac{1}{2} \quad ; \end{matrix}\ t^2\right] \tag{1.19.17}$$

$$\sin[m\{\sin^{-1}(t)\}] = m\ t\ {}_2F_1\left[\begin{matrix} \frac{1-m}{2}, \frac{1+m}{2}; \\ \frac{3}{2} \quad ; \end{matrix}\ t^2\right] \tag{1.19.18}$$

$$\cos[a\{\sin^{-1}(t)\}] = {}_2F_1\left[\begin{matrix} \frac{-a}{2}, \frac{a}{2}; \\ \frac{1}{2} \quad ; \end{matrix}\ t^2\right] \tag{1.19.19}$$

$$\sin[a\{\sin^{-1}(t)\}] = a\ t\ {}_2F_1\left[\begin{matrix} \frac{1-a}{2}, \frac{1+a}{2}; \\ \frac{3}{2} \quad ; \end{matrix}\ t^2\right] \tag{1.19.20}$$

$$\left[(1+t)^{2a} - (1-t)^{2a}\right] = 4a\ t\ {}_2F_1\left[\begin{matrix} 1-a, \frac{1}{2}-a; \\ \frac{3}{2} \quad ; \end{matrix}\ t^2\right] \qquad |t| < 1 \tag{1.19.21}$$

$$\left[(1+t)^{2a} + (1-t)^{2a}\right] = 2\ {}_2F_1\left[\begin{matrix} -a, \frac{1}{2}-a; \\ \frac{1}{2} \quad ; \end{matrix}\ t^2\right], \qquad |t| < 1 \tag{1.19.22}$$

$$ {}_2F_1\left[\begin{array}{cc} a, & a+\frac{1}{2}; \\ 2a+1 & ; \end{array} t\right] = \left(\frac{2}{1+\sqrt{(1-t)}}\right)^{2a} \qquad (1.19.23) $$

$$ {}_2F_1\left[\begin{array}{cc} a, & a+\frac{1}{2}; \\ 2a & ; \end{array} t\right] = \frac{1}{\sqrt{(1-t)}}\left(\frac{2}{1+\sqrt{(1-t)}}\right)^{2a-1} \qquad (1.19.24) $$

$$ \frac{1+t}{(1-t)^{2a+1}} = {}_2F_1\left[\begin{array}{c} 2a,\ a+1\ ; \\ a\ ; \end{array} t\right] \qquad |t|<1 \qquad (1.19.25) $$

$$ \left[1-\left(1-\frac{a}{c}\right)t\right](1-t)^{(-a-1)} = {}_2F_1\left[\begin{array}{c} a,\ c+1\ ; \\ c\ ; \end{array} t\right] \qquad |t|<1 \qquad (1.19.26) $$

$$ \left[\left(\sqrt{(1+t^2)}+t\right)^{2b} + \left(\sqrt{(1+t^2)}-t\right)^{2b}\right] = 2\ {}_2F_1\left[\begin{array}{c} b,\ -b\ ; \\ \frac{1}{2}\ ; \end{array} -t^2\right] \qquad (1.19.27) $$

$$ \frac{1}{\sqrt{(1+t^2)}}\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-1} + \left(\sqrt{(1+t^2)}-t\right)^{2b-1}\right] = 2\ {}_2F_1\left[\begin{array}{c} b,\ 1-b\ ; \\ \frac{1}{2}\ ; \end{array} -t^2\right] \qquad (1.19.28) $$

$$ \left[\left(\sqrt{(1+t^2)}+t\right)^{2b-1} - \left(\sqrt{(1+t^2)}-t\right)^{2b-1}\right] = 2\,(2b-1)\,t\ {}_2F_1\left[\begin{array}{c} b,\ 1-b\ ; \\ \frac{3}{2}\ ; \end{array} -t^2\right] \qquad (1.19.29) $$

$$ \frac{1}{\sqrt{(1+t^2)}}\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-2} - \left(\sqrt{(1+t^2)}-t\right)^{2b-2}\right] $$
$$ = 4\ (b-1)\ t\ {}_2F_1\left[\begin{array}{c} b,\ 2-b\ ; \\ \frac{3}{2}\ ; \end{array} -t^2\right] \qquad (1.19.30) $$

1.20 Hypergeometric Forms of Some Special Functions

For hypergeometric forms of Special functions of Mathematical Physics, we refer the monographs [1, 8, 9, 10, 24, 29, 38, 91, 92, 95, 119, 123, 182, 184, 185, 188, 196, 203, 213, 235, 252, 293, 294, 324, 326, 336, 337].

Ordinary Bessel Function of I Kind $J_\nu(t)$: [326, p.44 (11)]

$$ J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left[\begin{array}{c} -\ ; \\ \nu+1; \end{array} -\frac{t^2}{4}\right] \qquad (1.20.1) $$

Modified Bessel Function of I Kind $I_\nu(t)$: [326, p.44 (12)]

$$I_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left[\begin{array}{c} - \ ; \ \frac{t^2}{4} \\ \nu+1; \end{array}\right] \tag{1.20.2}$$

Error Function: [252, p.36 (6), p.127 Q.1; 119, p.887]

$$\mathfrak{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx = \frac{2t}{\sqrt{\pi}}\ {}_1F_1\left[\begin{array}{c} \frac{1}{2} \ ; \ -t^2 \\ \frac{3}{2} \ ; \end{array}\right], \qquad |t| < \infty \tag{1.20.3}$$

Incomplete Gamma Function: [252, p.127 Q.2; 184, p.220 (6.2.11.1)]

$$\gamma(a,t) = \int_0^t e^{-x}x^{a-1}dx = \frac{t^a}{a}\ {}_1F_1\left[\begin{array}{c} a \ ; \ -t \\ 1+a\ ; \end{array}\right], \qquad |arg(t)| < \pi, \qquad \Re(a) > 0 \tag{1.20.4}$$

Legendre's Normal Form of Incomplete Elliptic Integral of I Kind: [92, p.313 (13.6.1)]

$$F(k,\phi) = \int_0^\phi \frac{dt}{\sqrt{(1-k^2\sin^2(t))}}, \qquad |k| < 1 \tag{1.20.5}$$

Legendre's Normal Form of Incomplete Elliptic Integral of II Kind: [92, p.313 (13.6.2)]

$$E(k,\phi) = \int_0^\phi \sqrt{(1-k^2\sin^2(t))}\ dt, \qquad |k| < 1 \tag{1.20.6}$$

Legendre's Normal Form of Incomplete Elliptic Integral of III Kind: [92, p.313 (13.6.3)]

$$\Pi(\nu,k,\phi) = \int_0^\phi \frac{dt}{(1+\nu\sin^2(t))\sqrt{(1-k^2\sin^2(t))}}, \qquad |k| < 1 \tag{1.20.7}$$

where k is called modulus of the Elliptic integrals.

Complete Elliptic Integrals of I Kind: [92, p.317-318 (13.8.1)(13.8.5)]

$$\mathbf{K}(t) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-t^2\sin^2\theta)}} = \frac{\pi}{2}\ {}_2F_1\left[\begin{array}{c} \frac{1}{2}, \ \frac{1}{2}; \ t^2 \\ 1 \quad ; \end{array}\right], \qquad |t| < 1 \tag{1.20.8}$$

Complete Elliptic Integrals of II Kind: [92, p.317-318 (13.8.2)(13.8.6)]

$$\mathbf{E}(t) = \int_0^{\frac{\pi}{2}} \sqrt{(1-t^2\sin^2\theta)}d\theta = \frac{\pi}{2}\ {}_2F_1\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}; \ t^2 \\ 1 \quad ; \end{array}\right], \qquad |t| < 1 \tag{1.20.9}$$

Complete Elliptic Integrals of III Kind: [92, p.317 (13.8.3)]

$$\Pi_1(\nu,k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+\nu\sin^2\theta)\sqrt{(1-k^2\sin^2\theta)}}, \qquad |k|<1 \tag{1.20.10}$$

Complete Elliptic Integrals: [92, p.321 (13.8.25)]

$$\mathbf{B}(t) = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sqrt{(1-t^2\sin^2\theta)}}d\theta = \frac{\pi}{4}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}; \\ 2\ ;\end{matrix}\ t^2\right] \tag{1.20.11}$$

Complete Elliptic Integrals: [92, p.321 (13.8.25)]

$$\mathbf{C}(t) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\left(\sqrt{(1-t^2\sin^2\theta)}\right)^3}\,d\theta = \frac{\pi}{16}\,{}_2F_1\left[\begin{matrix}\frac{3}{2},\frac{3}{2}; \\ 3\ ;\end{matrix}\ t^2\right] \tag{1.20.12}$$

Complete Elliptic Integrals: [92, p.321 (13.8.25)]

$$\mathbf{D}(t) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\sqrt{(1-t^2\sin^2\theta)}}\,d\theta = \frac{\pi}{4}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{3}{2}; \\ 2\ ;\end{matrix}\ t^2\right] \tag{1.20.13}$$

Lerch's Transcendent:[188, p.32 (1.6); see also 119, p.1039]

$$\mathbf{\Phi}(t,q,a) = \sum_{n=0}^{\infty}(a+n)^{-q}t^n \tag{1.20.14}$$

($a \neq 0,-1,-2,\ldots$ and $|t|<1$).

When q is positive integer, then

$$a^q\mathbf{\Phi}(t,q,a) = {}_{q+1}F_q\left[\begin{matrix}1,\overbrace{a,a,\ldots,a}^{q}\ ; \\ \underbrace{a+1,a+1,\ldots,a+1}_{q};\end{matrix}\ t\right] \tag{1.20.15}$$

Here $\overbrace{a,a,\ldots,a}^{q}$ denotes the numerator parameter "a" is written "q" times and $\underbrace{a+1,a+1,\ldots,a+1}_{q}$ denotes the denominator parameter "$a+1$" is written "q" times.

Fresnel's Integral: [1, p.300 (7.3.2)]

$$\mathbf{S}(t) = \frac{\pi t^3}{6}\,{}_1F_2\left[\begin{matrix}\frac{3}{4}\ ; \\ \frac{3}{2},\frac{7}{4};\end{matrix}\ -\frac{\pi^2t^4}{16}\right] = \int_0^t \sin\left(\frac{\pi x^2}{2}\right)dx \tag{1.20.16}$$

Fresnel's Integral: [1, p.300 (7.3.4)]

$$\mathbf{S}_1(t) = \frac{t^3}{3}\sqrt{\frac{2}{\pi}}\,{}_1F_2\left[\begin{matrix}\frac{3}{4}\ ; \\ \frac{3}{2},\frac{7}{4};\end{matrix}\ -\frac{t^4}{4}\right] = \sqrt{\frac{2}{\pi}}\int_0^t \sin\left(x^2\right)dx \tag{1.20.17}$$

Fresnel's Integral: [1, p.300 (7.3.4)]

$$\mathbf{S}_2(t) = \frac{t}{3}\sqrt{\frac{2t}{\pi}}\ {}_1F_2\left[\begin{matrix} \frac{3}{4}\ ; & \\ \frac{3}{2}, \frac{7}{4}; & \end{matrix} -\frac{t^2}{4}\right] = \frac{1}{\sqrt{2\pi}}\int_0^t \frac{\sin(x)}{\sqrt{(x)}}dx \tag{1.20.18}$$

Fresnel's Integral: [1, p.300 (7.3.1)]

$$\mathbf{C}^*(t) = t_1F_2\left[\begin{matrix} \frac{1}{4}\ ; & \\ \frac{1}{2}, \frac{5}{4}; & \end{matrix} -\frac{\pi^2 t^4}{16}\right] = \int_0^t \cos\left(\frac{\pi x^2}{2}\right)dx \tag{1.20.19}$$

Fresnel's Integral: [1, p.300 (7.3.3)]

$$\mathbf{C}_1(t) = t\sqrt{\frac{2}{\pi}}\ {}_1F_2\left[\begin{matrix} \frac{1}{4}\ ; & \\ \frac{1}{2}, \frac{5}{4}; & \end{matrix} -\frac{t^4}{4}\right] = \sqrt{\frac{2}{\pi}}\int_0^t \cos\left(x^2\right)dx \tag{1.20.20}$$

Fresnel's Integral: [1, p.300 (7.3.3)]

$$\mathbf{C}_2(t) = \sqrt{\frac{2t}{\pi}}\ {}_1F_2\left[\begin{matrix} \frac{1}{4}\ ; & \\ \frac{1}{2}, \frac{5}{4}; & \end{matrix} -\frac{t^2}{4}\right] = \frac{1}{\sqrt{2\pi}}\int_0^t \frac{\cos(x)}{\sqrt{(x)}}dx \tag{1.20.21}$$

Sine Integrals: [119, p.886 (8.230)]

$$S_i(t) = \int_0^t \frac{\sin(x)}{x}dx = t\ {}_1F_2\left[\begin{matrix} \frac{1}{2}, & & ; \\ \frac{3}{2}, & \frac{3}{2} & ; \end{matrix} \frac{-t^2}{4}\right] \tag{1.20.22}$$

Hyperbolic Sine Integrals: [119, p.886 (8.221)]

$$Shi(t) = \int_0^t \frac{\sinh(x)}{x}dx = t\ {}_1F_2\left[\begin{matrix} \frac{1}{2}, & & ; \\ \frac{3}{2}, & \frac{3}{2} & ; \end{matrix} \frac{t^2}{4}\right] \tag{1.20.23}$$

where $t > 0$, or $|arg(t)| < \pi$.

Polylogarithm Functions: [91, p.30 (1.11.14)]

$$\mathbf{Li}_q(t) = \sum_{k=1}^{\infty}\frac{t^k}{k^q} \tag{1.20.24}$$

$$\mathbf{Li}_q(t) = t\ {}_{q+1}F_q\left[\begin{matrix} \overbrace{1,1,1,1,\ldots 1}^{\text{q+1}} & ; \\ \underbrace{2,2,2,\ldots\ldots 2}_{\text{q}} & ; \end{matrix}\ t\right], \qquad |t| \leq 1 \tag{1.20.25}$$

Here $\overbrace{1,1,1,1,\ldots 1}^{\text{q+1}}$ denotes the numerator parameter "1" is written "$q+1$" times and $\underbrace{2,2,2,\ldots\ldots 2}_{\text{q}}$ denotes the denominator parameter "2" is written "q" times, where $q = 2, 3, 4, \ldots$. When $q = 2$ it is called Dilogarithm function.

Struve Functions: [1, p.496 (12.1.3)]

$$\mathbf{H}_v(t) = \frac{(\frac{t}{2})^{\nu+1}}{\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})}\ {}_1F_2\left[\begin{array}{ccc} 1 & ; & \frac{-t^2}{4} \\ \frac{3}{2}, \nu+\frac{3}{2}; & & \end{array}\right] \tag{1.20.26}$$

Modified Struve Functions: [1, p.498 (12.2.1)]

$$\mathbf{L}_v(t) = \frac{(\frac{t}{2})^{\nu+1}}{\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})}\ {}_1F_2\left[\begin{array}{ccc} 1 & ; & \frac{t^2}{4} \\ \frac{3}{2}, \nu+\frac{3}{2}; & & \end{array}\right] \tag{1.20.27}$$

Lommel Functions: [184, p.217 (6.2.9.1)]

$$\mathbf{s}_{\mu,\nu}(t) = \frac{(t)^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)}\ {}_1F_2\left[\begin{array}{ccc} 1 & ; & \frac{-t^2}{4} \\ \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; & & \end{array}\right] \tag{1.20.28}$$

where $\mu \pm \nu \neq -1, -3, -5, \ldots$

Kelvin's Functions: [119, p.944 (8.564(1))]

$$\mathfrak{ber}(t) = {}_0F_3\left[\begin{array}{ccc} - & ; & -\frac{t^4}{256} \\ \frac{1}{2}, \quad \frac{1}{2}, \quad 1; & & \end{array}\right] \quad ; \quad t \in R \tag{1.20.29}$$

Kelvin's Functions: [119, p.944 (8.564(2))]

$$\mathfrak{bei}(t) = \frac{t^2}{4}\ {}_0F_3\left[\begin{array}{ccc} - & ; & -\frac{t^4}{256} \\ \frac{3}{2}, \quad \frac{3}{2}, \quad 1; & & \end{array}\right] \quad ; \quad t \in R \tag{1.20.30}$$

The Incomplete Beta Function: [326, p.35 (31)]

$$\mathbf{B}_t(\alpha, \beta) = \int_0^t x^{\alpha-1}(1-x)^{\beta-1}dx = \alpha^{-1}\ t^{\alpha}\ {}_2F_1\left[\begin{array}{ccc} \alpha, & 1-\beta; & t \\ 1+\alpha & ; & \end{array}\right] \tag{1.20.31}$$

The product $J_\mu(t)J_\nu(t)$ is given in the [184, p.216 (39)]

$$J_\mu(t)J_\nu(t) = \frac{\left(\frac{t}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\ {}_2F_3\left[\begin{array}{cc} \frac{\mu+\nu+1}{2}, \quad \frac{\mu+\nu+2}{2}; & -t^2 \\ \mu+1, \ \nu+1, \ \mu+\nu+1; & \end{array}\right] \tag{1.20.32}$$

The product $J_\nu(t)J_{\nu+1}(t)$ is given in the [184, p.216 (40)]

$$J_\nu(t)J_{\nu+1}(t) = \frac{\left(\frac{t}{2}\right)^{2\nu+1}}{\Gamma(\nu+1)\Gamma(\nu+2)}\ {}_1F_2\left[\begin{array}{cc} \nu+\frac{3}{2}\ ; & -t^2 \\ \nu+2, \ 2\nu+2; & \end{array}\right] \tag{1.20.33}$$

The product $J_\nu^2(t)$ is given in the [184, p.216 (41)]

$$J_\nu^2(t) = \frac{\left(\frac{t}{2}\right)^{2\nu}}{[\Gamma(\nu+1)]^2} \; {}_1F_2\left[\begin{matrix} \nu+\frac{1}{2}\;; \\ \nu+1,\; 2\nu+1; \end{matrix} \; -t^2\right] \tag{1.20.34}$$

The product $J_{-\nu}(t)J_\nu(t)$ is given in the [184, p.216 (42)]

$$J_{-\nu}(t)J_\nu(t) = \frac{\sin(\nu\pi)}{\nu\pi} \; {}_1F_2\left[\begin{matrix} \frac{1}{2} \;; \\ 1+\nu, 1-\nu; \end{matrix} \; -t^2\right] \tag{1.20.35}$$

The product $J_\nu(t)I_\nu(t)$ is given in the [184, p.216 (43)]

$$J_\nu(t)I_\nu(t) = \frac{\left(\frac{t}{2}\right)^{2\nu}}{[\Gamma(\nu+1)]^2} \; {}_0F_3\left[\begin{matrix} - \;; \\ \frac{\nu+1}{2}, \frac{\nu+2}{2}, \nu+1; \end{matrix} \; -\frac{t^4}{64}\right] \tag{1.20.36}$$

Hyper-Bessel Function of Humbert P.: [95, p.250 (19.7.7)(19.7.8); see also [135]; 219, p.102]

$$J_{m,n}(z) = \frac{\left(\frac{z}{3}\right)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} \; {}_0F_2\left[\begin{matrix} - \;; \\ m+1,\; n+1\;; \end{matrix} \; -\frac{z^3}{27}\right] \tag{1.20.37}$$

where m, n are may be positive and negative integer.

Modified Hyper-Bessel Function by Delerue [76]

$$I_{m,n}(z) = \frac{\left(\frac{z}{3}\right)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} \; {}_0F_2\left[\begin{matrix} - \;; \\ m+1,\; n+1\;; \end{matrix} \; \frac{z^3}{27}\right] \tag{1.20.38}$$

Arctangent Function

$$\begin{aligned} Ti_2(t) &= \int_0^t \frac{\tan^{-1}(x)}{x}dx = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)^2} \\ &= t\; {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1; \\ \frac{3}{2}, \quad \frac{3}{2}; \end{matrix} \; -t^2\right] \end{aligned} \tag{1.20.39}$$

1.21 Transformation and Reduction Formulas

For transformations and reduction formulas of multiple hypergeometric functions and product theorems of hypergeometric functions, we refer the monographs and research papers [8, 17, 91, 124, 151, 184, 235, 252, 293, 326].

Pfaff Kummer Linear Transformation:

$${}_2F_1\left[\begin{matrix} a, b; \\ c \;; \end{matrix} \; t\right] = (1-t)^{(-a)} {}_2F_1\left[\begin{matrix} a, c-b; \\ c \;; \end{matrix} \; \frac{-t}{1-t}\right] \tag{1.21.1}$$

$c \neq 0, -1, -2, \ldots, \qquad |arg(1-t) < \pi|$

Euler's Linear Transformation:

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c\ ; \end{matrix}\ t\right] = (1-t)^{(c-a-b)} {}_2F_1\left[\begin{matrix} c-a, c-b; \\ c\quad ; \end{matrix}\ t\right] \tag{1.21.2} $$

$c \neq 0, -1, -2, \ldots, \qquad |arg(1-t) < \pi|$

Kummer's I Transformation:

$$ {}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix}\ t\right] = e^t {}_1F_1\left[\begin{matrix} c-a; \\ c\quad ; \end{matrix}\ -t\right] \tag{1.21.3} $$

where $c \neq 0, -1, -2, \ldots$

Kummer's II Transformation:

$$ {}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\ 2t\right] = e^t {}_0F_1\left[\begin{matrix} - & ; \\ a+\frac{1}{2} & ; \end{matrix}\ \frac{t^2}{4}\right] \tag{1.21.4} $$

where $2a$ is not an odd integer < 0 i.e. $2a \neq -1, -3, -5, \ldots$

A transformation [3, p.128 (3.1.11)] is given by

$$ \frac{1}{(1+t)^{2a}}\ {}_2F_1\left[\begin{matrix} a, b; \\ 2b\ ; \end{matrix}\ \frac{4t}{(1+t)^2}\right] = {}_2F_1\left[\begin{matrix} a, a+\frac{1}{2}-b; \\ b+\frac{1}{2}\qquad ; \end{matrix}\ t^2\right] \tag{1.21.5} $$

A transformation [3, p.125 (3.1.3)] is given by

$$ {}_2F_1\left[\begin{matrix} a, & b; \\ a+b+\frac{1}{2}; & \end{matrix}\ 4t(1-t)\right] = {}_2F_1\left[\begin{matrix} 2a, & 2b; \\ a+b+\frac{1}{2}; & \end{matrix}\ t\right] \tag{1.21.6} $$

where $a+b+\frac{1}{2} \neq 0, -1, -2, \ldots$ and $\Re(t) < \frac{1}{2}$

Goursat's Quadratic Transformation: [91, p.113, (1.11.32, 1.11.34)]

$$ {}_2F_1[a, b; a-b+1; z] = (1+z)^{-a} {}_2F_1\left[\frac{a}{2}, \frac{a+1}{2}; a-b+1; 4z(1+z)^{-2}\right] \tag{1.21.7} $$

A transformation [3, p.127 (3.1.3)] is given by

$$ \left(1-\frac{t}{2}\right)^{-b} {}_2F_1\left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2}; \\ a+\frac{1}{2} & ; \end{matrix}\ \left(\frac{t}{2-t}\right)^2\right] = {}_2F_1\left[\begin{matrix} a, & b; \\ 2a & ; \end{matrix}\ t\right] \tag{1.21.8} $$

Clausen's identity [3, p.116 Q.13] is given by

$$ \left({}_2F_1\left[\begin{matrix} a, & b\ ; \\ a+b+\frac{1}{2}; & \end{matrix}\ t\right]\right)^2 = {}_3F_2\left[\begin{matrix} 2a, & 2b, & a+b\ ; \\ 2a+2b, & a+b+\frac{1}{2}; & \end{matrix}\ t\right] \tag{1.21.9} $$

A transformation [3, p.181 Q.20] is given

$$\left({}_2F_1 \left[\begin{matrix} a, & b; \\ c & ; \end{matrix} \; t \right] \right)^2$$

$$= \sum_{n=0}^{\infty} \frac{(2a)_n (2b)_n (c-\frac{1}{2})_n}{(c)_n \; (2c-1)_n \; n!} \; {}_4F_3 \left[\begin{matrix} \frac{-n}{2}, & \frac{-n+1}{2}, & \frac{1}{2}, & a+b+\frac{1}{2}-c; \\ a+\frac{1}{2}, & b+\frac{1}{2}, & \frac{3}{2}-n-c & ; \end{matrix} \; 1 \right] t^n \tag{1.21.10}$$

A transformation [3, p.116 Q.13; see also 17, p.100 Q.16] is given by

$${}_2F_1 \left[\begin{matrix} a, & b \; ; \\ a+b+\frac{1}{2}; & \end{matrix} \; t \right] {}_2F_1 \left[\begin{matrix} \frac{1}{2}-a, & \frac{1}{2}-b \; ; \\ \frac{3}{2}-a-b & ; \end{matrix} \; t \right] = {}_3F_2 \left[\begin{matrix} a-b+\frac{1}{2}, & b-a+\frac{1}{2}, & \frac{1}{2}; \\ a+b+\frac{1}{2}, & \frac{3}{2}-a-b & ; \end{matrix} \; t \right] \tag{1.21.11}$$

A transformation [3, p.184 Q.31(a),31(b); see also 293, p.77 (2.5.12 ; 2.5.13); 17, p.86 (5;6)] is given by

$${}_2F_1 \left[\begin{matrix} a, & b \; ; \\ a+b-\frac{1}{2} \; ; & \end{matrix} \; t \right] {}_2F_1 \left[\begin{matrix} a, & b \; ; \\ a+b+\frac{1}{2} \; ; & \end{matrix} \; t \right] = {}_3F_2 \left[\begin{matrix} 2a, & 2b, & a+b \; ; \\ 2a+2b-1, & a+b+\frac{1}{2}; & \end{matrix} \; t \right] \tag{1.21.12}$$

$${}_2F_1 \left[\begin{matrix} a, & b \; ; \\ a+b-\frac{1}{2} \; ; & \end{matrix} \; t \right] {}_2F_1 \left[\begin{matrix} a, & b-1; \\ a+b-\frac{1}{2} & ; \end{matrix} \; t \right] = {}_3F_2 \left[\begin{matrix} 2a, & 2b-1, & a+b-1; \\ 2a+2b-2 \; , & a+b-\frac{1}{2} & ; \end{matrix} \; t \right] \tag{1.21.13}$$

Reduction Formulas for Appell's Functions:

$$\mathbf{F}_1[a; b, c; d; t, t] = {}_2F_1 \left[\begin{matrix} a, & b+c; \\ & d \;\; ; \end{matrix} \; t \right] \tag{1.21.14}$$

$$\mathbf{F}_1[a; b, b; c; t, -t] = {}_3F_2 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, b \; ; \\ \frac{c}{2}, \quad \frac{c+1}{2}; \end{matrix} \; t^2 \right] \tag{1.21.15}$$

$$\mathbf{F}_2[a; b, b; c, c; t, -t] = {}_4F_3 \left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & b, & c-b; \\ c, & \frac{c}{2}, & \frac{c+1}{2} & ; \end{matrix} \; t^2 \right], \qquad |t| < \frac{1}{2} \tag{1.21.16}$$

$$\mathbf{F}_2[a; \lambda, \mu; 2\lambda, 2\mu; t, -t] = {}_4F_3 \left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & \frac{\lambda+\mu}{2}, & \frac{\lambda+\mu+1}{2}; \\ \lambda+\mu, & \lambda+\frac{1}{2}, & \mu+\frac{1}{2} & ; \end{matrix} \; t^2 \right], \qquad |t| < \frac{1}{2} \tag{1.21.17}$$

$$\mathbf{F}_3[a, a; b, b; c; t, -t] = {}_4F_3 \left[\begin{matrix} a, & b, & \frac{a+b}{2}, & \frac{a+b+1}{2}; \\ a+b, & \frac{c}{2}, & \frac{c+1}{2} & ; \end{matrix} \; t^2 \right], \qquad |t| < 1 \tag{1.21.18}$$

$$\mathbf{F}_4[a,b;c,d;t,t] = {}_4F_3\left[\begin{matrix} a, & b, & \frac{c+d}{2}, & \frac{c+d-1}{2}; \\ c, & d, & c+d-1 & ; \end{matrix} \; 4t\right] \tag{1.21.19}$$

where $\sqrt{|t|} < \frac{1}{2}$.

$$\mathbf{F}_4[a,b;c,c;t,-t] = {}_4F_3\left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & \frac{b}{2}, & \frac{b+1}{2}; \\ c, & \frac{c}{2}, & \frac{c+1}{2} & ; \end{matrix} \; -4t^2\right], \tag{1.21.20}$$

where $\sqrt{|t|} < \frac{1}{2}$.

Reduction Formula of Bailey: [17, p.102 Q.20(v)]

$$F_4\left[\alpha,\beta;\alpha-\beta+1,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right]$$
$$= (1-y)^{\alpha}{}_2F_1\left[\alpha,\beta;\alpha-\beta+1;-\frac{x(1-y)}{(1-x)}\right] \tag{1.21.21}$$

Ramanujan's Theorem 1:

$${}_1F_1\left[\begin{matrix} a\ ; \\ b\ ; \end{matrix}\; t\right]{}_1F_1\left[\begin{matrix} a\ ; \\ b\ ; \end{matrix}\; -t\right] = {}_2F_3\left[\begin{matrix} a\ , & b-a & ; \\ b\ , & \frac{b}{2}, & \frac{b+1}{2} & ; \end{matrix}\; \frac{t^2}{4}\right] \tag{1.21.22}$$

Ramanujan's Theorem 2:

$${}_0F_2\left[\begin{matrix} - & ; \\ a,\ b & ; \end{matrix}\; t\right]{}_0F_2\left[\begin{matrix} - & ; \\ a,\ b & ; \end{matrix}\; -t\right]$$

$$= {}_3F_8\left[\begin{matrix} \frac{a+b-1}{3}, & \frac{a+b}{3}, & \frac{a+b+1}{3} & ; \\ a\ ,\ b\ ,\ \frac{a}{2},\ \frac{b}{2}\ ,\ \frac{a+1}{2}, & \frac{b+1}{2}, & \frac{a+b-1}{2},\ \frac{a+b}{2} & ; \end{matrix}\; -\frac{27t^2}{64}\right] \tag{1.21.23}$$

$${}_1F_1\left[\begin{matrix} a\ ; \\ 2a\ ; \end{matrix}\; t\right]{}_1F_1\left[\begin{matrix} b\ ; \\ 2b\ ; \end{matrix}\; -t\right] = {}_2F_3\left[\begin{matrix} \frac{a+b}{2}, & \frac{a+b+1}{2} & ; \\ a+\frac{1}{2}\ , & b+\frac{1}{2}, \quad a+b & ; \end{matrix}\; \frac{t^2}{4}\right] \tag{1.21.24}$$

Whipple Quadratic Transformation: [8, p.130, (3.1.15)]

$${}_3F_2\left[\begin{matrix} a,\ b & ,\ c & ; \\ 1+a-b, & 1+a-c; \end{matrix}\; t\right] = (1-t)^{-a}{}_3F_2\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}\ , & 1+a-b-c; \\ 1+a-b\ , & 1+a-c\ ; \end{matrix}\; -\frac{4t}{(1-t)^2}\right] \tag{1.21.25}$$

Henrici's Triple Product Theorem: [324, p.85 (1); see also 124]

$${}_0F_1\left[\begin{matrix} -\ ; \\ 6c\ ; \end{matrix}\; t\right]{}_0F_1\left[\begin{matrix} -\ ; \\ 6c\ ; \end{matrix}\; \omega t\right]{}_0F_1\left[\begin{matrix} -\ ; \\ 6c\ ; \end{matrix}\; \omega^2 t\right]$$

$$= {}_2F_7\left[\begin{matrix} 3c-\frac{1}{4} \;,\; 3c+\frac{1}{4} \;; \\ 6c, 2c,\; 2c+\frac{1}{3}\;,\; 2c+\frac{2}{3}\;,\; 4c-\frac{1}{3}\;,\; 4c\;,\; 4c+\frac{1}{3}\;; \end{matrix}\; \left(\frac{4t}{9}\right)^3\right] \tag{1.21.26}$$

where $\omega = exp\left(\frac{2\pi i}{3}\right)$

$${}_0F_1\left[\begin{matrix} -; \\ a\;; \end{matrix}\; t\right] {}_0F_1\left[\begin{matrix} -; \\ a\;; \end{matrix}\; -t\right] = {}_0F_3\left[\begin{matrix} - \;; \\ a,\; \frac{a}{2},\; \frac{a+1}{2}; \end{matrix}\; -\frac{t^2}{4}\right] \tag{1.21.27}$$

$${}_0F_1\left[\begin{matrix} -; \\ a\;; \end{matrix}\; t\right] {}_0F_1\left[\begin{matrix} -; \\ b\;; \end{matrix}\; t\right] = {}_2F_3\left[\begin{matrix} \frac{a+b}{2},\; \frac{a+b-1}{2} \;; \\ a,\; b,\; a+b-1; \end{matrix}\; 4t\right] \tag{1.21.28}$$

Bailey Cubic Transformation: [8, P.185, Q.38(a); Q.38(b)]

$${}_3F_2\left[\begin{matrix} a,\; 2b-a-1\;,\; a+2-2b\;; \\ b,\; a-b+\frac{3}{2}\;; \end{matrix}\; \frac{t}{4}\right] = (1-t)^{-a}{}_3F_2\left[\begin{matrix} \frac{a}{3},\; \frac{a+1}{3}\;,\; \frac{a+2}{3}; \\ b\;,\; a-b+\frac{3}{2}\;; \end{matrix}\; -\frac{27t}{4(1-t)^3}\right] \tag{1.21.29}$$

$${}_3F_2\left[\begin{matrix} a,\; b-\frac{1}{2}\;,\; 1+a-b\;; \\ 2b,\; 2+2a-2b\;; \end{matrix}\; t\right] = \left(1-\frac{t}{4}\right)^{-a}{}_3F_2\left[\begin{matrix} \frac{a}{3},\; \frac{a+1}{3}\;,\; \frac{a+2}{3}; \\ b\;,\; a-b+\frac{3}{2}\;; \end{matrix}\; \frac{27t^2}{(4-t)^3}\right] \tag{1.21.30}$$

$$\left({}_1F_1\left[\begin{matrix} a\;; \\ 2a\;; \end{matrix}\; t\right]\right)^2 = e^t{}_1F_2\left[\begin{matrix} a\;; \\ a+\frac{1}{2}\;,\; 2a\;; \end{matrix}\; \frac{t^2}{4}\right] \tag{1.21.31}$$

1.22 Multiple Series Identities

Series Rearrangement Technique:

The technique described and illustrated in the thesis is based, in part, upon certain interchanges of the order of a double (or multiple) summation. Such rearrangement of terms in iterated finite series can be justified in the elementary sense when, for example, the series involved are absolutely convergent. Thus the identities contained in lemmas 1, 2, 3, 4, 5 and 6 below may be considered as purely formal.

LEMMA 1

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{1.22.1}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k) \tag{1.22.2}$$

LEMMA 2

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(k,n-2k) \tag{1.22.3}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+2k) \tag{1.22.4}$$

where, and in what follows, $[x]$ denotes the greatest integer in x.

LEMMA 3 For any positive integer m,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} A(k,n-mk) \tag{1.22.5}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+mk) \tag{1.22.6}$$

For $m = 1$, Lemma 3 evidently reduces to Lemma 1, while a special case of Lemma 3 when $m = 2$ is precisely Lemma 2. These special cases of Lemma 3 were stated and proved by [252, p.56 sec. 37].

LEMMA 4 Multiple Series Identities: [326, p.102 (16);(17)]

$$\sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\phi(n;k_1,k_2,\ldots,k_m)\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,\ldots,k_m=0}^{k_1+k_2+\ldots+k_m\leq n}\phi(n-k_1-k_2-\ldots-k_m;k_1,k_2,\ldots,k_m)\right) \tag{1.22.7}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,\ldots,k_m=0}^{k_1+k_2+\ldots+k_m\leq n}\phi(n;k_1,k_2,\ldots,k_m)\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\phi(n+k_1+k_2+\ldots+k_m;k_1,k_2,\ldots,k_m)\right) \tag{1.22.8}$$

LEMMA 5 Finite Double Series Identity: [326, p.103, (2.1.18)]

$$\sum_{n=0}^{p}\sum_{k=0}^{[\frac{n}{m}]} B(k,n) = \sum_{n=0}^{p}\sum_{k=0}^{[\frac{p-n}{m}]} B(k,n+mk) \tag{1.22.9}$$

where, and in what follows, $[x]$ denotes the greatest integer in x.

$$\sum_{r=0}^{m}\sum_{s=0}^{r} B(s,r) = \sum_{r=0}^{m}\sum_{s=0}^{m-r} B(s,s+r) \tag{1.22.10}$$

LEMMA 6 Series Identity: [326, p.214 Q.8]

$$\sum_{m,n=0}^{\infty} \Psi(m,n) = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \sum_{m,n=0}^{\infty} \Psi(mM+j, nN+k) \tag{1.22.11}$$

where $\forall M, N \in \{1, 2, 3, \ldots\}$

$$\sum_{m=0}^{\infty} \sum_{q=0}^{\infty} A(m,q) = \sum_{j=0}^{2n-1} \sum_{i=0}^{2-1} \sum_{m,q=0}^{\infty} A(2nm+j, 2q+i) \tag{1.22.12}$$

Decomposition Technique:

$$\sum_{m=0}^{\infty} \Phi(m) = \sum_{m=0}^{\infty} \Phi(2m) + \sum_{m=0}^{\infty} \Phi(2m+1) \tag{1.22.13}$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(m,n) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m,2n) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m,2n+1)$$
$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m+1,2n) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m+1,2n+1) \tag{1.22.14}$$

1.23 Some Hypergeometric Polynomials, their Relationships and Generating Relations

Special functions play a vital role in both classical as well as quantum physics. There are many special functions, in which one of the most convenient depends on the particular problem at hand. Classical orthogonal polynomials play a central role in optics and certain parts of quantum mechanics.

Srivastava's General Class of Polynomials:

We begin by recalling the definition of the general class of polynomials $S_n^m(x)$ introduced by Srivastava

$$S_n^m(x) = \sum_{k=0}^{\frac{n}{m}} \frac{(-n)_{mk}}{k!} \, A_{n,k} \, x^k \qquad (n = 0, 1, 2, 3, \ldots) \tag{1.23.1}$$

where m is an arbitrary positive integer, the coefficient $A_{n,k}$ $(n, k \geq 0)$ are arbitrary constants, real or complex.

By suitably specializing the coefficient $A_{n,k}$, the Srivastava's generalized polynomials $S_n^m(x)$ can easily be reduced to the classical orthogonal polynomials including, for example, the Hermite polynomials $H_n(x)$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, and the Laguerre polynomials $L_n^{(\alpha)}(x)$ and indeed also to several familiar particular cases of the Jacobi polynomials such as the Gegenbauer (or Ultraspherical) polynomials $C_n^{\nu}(x)$, the Legendre polynomials $P_n(x)$, and the Tchebycheff polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds.

Other interesting special cases of the polynomials $S_n^m(x)$ include such generalized hypergeometric polynomials as the Bessel polynomials $y_n(x,\alpha,\beta)$ considered by Krall and Frink [175, p.108 (34)], the generalized Hermite polynomials $g_n^m(x,h)$ considered by Gould and Hopper [118, p.58], and the Brafman polynomials [28, p.186] which contain $g_n^m(x,h)$ as a particular case. Furthermore, since

$$H_{n,m,\nu}(x) = \nu^n \ g_n^m\left(x, -\frac{1}{\nu^m}\right) = g_n^m(\nu \ x, -1) \tag{1.23.2}$$

the Gould-Hopper polynomials $g_n^m(x,h)$ contain, as a special case, the generalized Hermite polynomials $H_{n,m,\nu}(x)$ considered by Lahiri[177, p.118 (3.2)].

Classical Hermite Polynomial:

$$H_n(x) = g_n^2(2x,-1) = H_n^{(2)}(2x,-1) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}$$

$$H_n(x) = (2x)^n {}_2F_0\left[\begin{array}{c} -\frac{n}{2}, \frac{-n+1}{2}; \\ -; \end{array} -\frac{1}{x^2}\right]$$

Hermite polynomials $H_n(x)$ are given by means of the generating relation

$$exp(2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{1.23.3}$$

valid for all finite x and t.

$$H_{2n}(x) = (2x)^{2n} {}_2F_0\left[\begin{array}{c} -n, \ \frac{1}{2}-n \ ; \\ - \qquad ; \end{array} -\frac{1}{x^2}\right] \tag{1.23.4}$$

$$H_{2n+1}(x) = (2x)^{2n+1} {}_2F_0\left[\begin{array}{c} -n, -\frac{1}{2}-n; \\ - \qquad ; \end{array} -\frac{1}{x^2}\right] \tag{1.23.5}$$

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{\left(-\frac{1}{2}\right)}(x^2) \tag{1.23.6}$$

$$H_{2n}(x)=\frac{(-1)^n(2n)!}{n!}{}_1F_1\left[\begin{matrix}-n; & \\ \frac{1}{2}\ ; & \end{matrix} x^2\right] \tag{1.23.7}$$

$$H_{2n+1}(x)=(-1)^n 2^{2n+1} n!\ (x)\ L_n^{(\frac{1}{2})}(x^2) \tag{1.23.8}$$

$$H_{2n+1}(x)=\frac{(-1)^n(2n+1)!}{n!}2x{}_1F_1\left[\begin{matrix}-n; & \\ \frac{3}{2}\ ; & \end{matrix} x^2\right] \tag{1.23.9}$$

A generating relation [95, p.250 (19.7.11)] is given by

$$(1+4t^2)^{\frac{-3}{2}}(1+2xt+4t^2)exp\left(\frac{4x^2t^2}{1+4t^2}\right)=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{[\frac{n}{2}]!} \tag{1.23.10}$$

$$\left[\frac{n}{2}\right]=\begin{cases}\frac{n}{2}, & \text{if } n \text{ is even}\\ \frac{n-1}{2}, & \text{if } n \text{ is odd}\end{cases} \tag{1.23.11}$$

A generating relation [95, p.263 (19.9.15)] is given by

$$(1+4t^2)^{-c}{}_1F_1\left[\begin{matrix}c; & \frac{4x^2t^2}{1+4t^2}\\ \frac{1}{2}; & \end{matrix}\right]+\frac{32ct^3x^3}{3(1+4t^2)^{c+2}}{}_1F_1\left[\begin{matrix}c+1\ ; & \frac{4x^2t^2}{1+4t^2}\\ \frac{5}{2}\qquad ; & \end{matrix}\right]$$

$$+\frac{2xt(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}}{}_1F_1\left[\begin{matrix}c; & \frac{4x^2t^2}{1+4t^2}\\ \frac{3}{2}; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(c)_l 2^{2l}H_n(x)t^n}{(2l)!} \tag{1.23.12}$$

$$l=\left[\frac{n}{2}\right]=\begin{cases}\frac{n}{2}, & \text{if } n \text{ is even integer}\\ \frac{n-1}{2}, & \text{if } n \text{ is odd integer}\end{cases} \tag{1.23.13}$$

Mehler's formula [326, P.83 (13)] is given by

$$\sum_{n=0}^{\infty}H_n(x)H_n(y)=(1-4t^2)^{-\frac{1}{2}}exp\left(\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right) \tag{1.23.14}$$

$$\sum_{n=0}^{\infty}H_n(x)H_n(y)=(1-4t^2)^{-\frac{1}{2}}exp\left(\frac{4xyt}{1-4t^2}\right)exp\left(\frac{-4(x^2+y^2)t^2}{1-4t^2}\right) \tag{1.23.15}$$

A generating relation [252, p.198 (2)] is given by

$$\sum_{n=0}^{\infty}H_n(x)H_n(y)=(1-4t^2)^{-\frac{1}{2}}exp\left[y^2-\frac{(y-2xt)^2}{1-4t^2}\right] \tag{1.23.16}$$

Weber-Hermite Function of Order n:

$$\Psi_n(x)=e^{-\frac{x^2}{2}}H_n(x) \tag{1.23.17}$$

$$H_n(x)=2^{\frac{n}{2}}e^{\frac{x^2}{2}}D_n(\sqrt{2}\ x) \tag{1.23.18}$$

$$He_n(x) = 2^{(-\frac{n}{2})} H_n\left(\frac{x}{\sqrt{2}}\right) \tag{1.23.19}$$

$$He_n(x) = e^{\frac{x^2}{4}} D_n(x) \tag{1.23.20}$$

where $D_n(x)$ is called Parabolic cylinder function.

Generalized Laguerre Polynomials of One, Two, Three and More Variable of Khan-Shukla:

Generalized Laguerre polynomials of one variable $L_n^{(\alpha)}(x)$ possess the following generating relation [252, p.130]

$$e^t {}_0F_1(-;1+\alpha;-xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) t^n}{(1+\alpha)_n} \tag{1.23.21}$$

Laguerre Polynomial [326, p.131 (2.5.1)]

$$L_n^{(\alpha)}(x) = \lim_{|\beta| \longrightarrow \infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)\right\} \tag{1.23.22}$$

A generating relation [95, p.249 (19.7.1)] is given by

$$(t-1)^m e^{xt} = \sum_{n=-m}^{\infty} x^m L_m^{(n)}(x) m! \frac{t^{m+n}}{(m+n)!} \tag{1.23.23}$$

Generalization of the generating function [326, p.250; see also] is given by

$$\sum_{n=0}^{\infty} L_m^{(n-m)}(x) \frac{t^n}{n!} = \frac{(t-x)^m}{m!} e^t \tag{1.23.24}$$

Laguerre Polynomials of Two Variables: [250]

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \psi_2\left[-n; 1+\alpha, 1+\beta; x, y\right] \tag{1.23.25}$$

$$\psi_2\left[\alpha; \gamma, \gamma'; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m y^n}{(m)!(n)!} \tag{1.23.26}$$

$$(|x| < \infty, \quad |y| < \infty)$$

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}}{(1+\alpha)_s (1+\beta)_r} \frac{x^s y^r}{(r)!(s)!}$$

$$= \frac{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{(n!)} \sum_{r=0}^{n} \frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r!\Gamma(1+\alpha+n-r)\Gamma(1+\beta+r)} \tag{1.23.27}$$

Chatterjea [47] gave generating function of Laguerre polynomials of two variable $L_n^{(\alpha,\beta)}(x,y)$ in the form

$$e^t {}_0F_1(-;1+\alpha;-xt){}_0F_1(-;1+\beta;-yt) = \sum_{n=0}^{\infty} \frac{n!\ L_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} \tag{1.23.28}$$

$$(1-t)^{-c}\psi_2\left[c;1+\alpha,1+\beta;\frac{-xt}{(1-t)},\frac{-yt}{(1-t)}\right] = \sum_{n=0}^{\infty} \frac{(n!)(c)_n\ L_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} \tag{1.23.29}$$

$$L_n^{(\alpha,0)}(x,0) = L_n^{(\alpha)}(x) \tag{1.23.30}$$

$$L_n^{(0,\beta)}(0,y) = L_n^{(\beta)}(y) \tag{1.23.31}$$

Laguerre Polynomials of Three Variables: [166]

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)$$
$$= \frac{(1+\alpha)_n(1+\beta)_n(1+\gamma)_n}{(n!)^3} \sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}}{(1+\alpha)_k(1+\beta)_s(1+\gamma)_r}\frac{x^k}{k!}\frac{y^s}{s!}\frac{z^r}{r!} \tag{1.23.32}$$

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z) = \frac{(1+\alpha)_n(1+\beta)_n(1+\gamma)_n}{(n!)^3}\psi_2^{(3)}\left[-n;1+\alpha,1+\beta,1+\gamma;x,y,z\right] \tag{1.23.33}$$

$$e^t {}_0F_1(-;1+\alpha;-xt){}_0F_1(-;1+\beta;-yt){}_0F_1(-;1+\gamma;-zt) = \sum_{n=0}^{\infty} \frac{(n!)^2\ L_n^{(\alpha,\beta,\gamma)}(x,y,z)t^n}{(1+\alpha)_n(1+\beta)_n(1+\gamma)_n} \tag{1.23.34}$$

$$(1-t)^{-c}\psi_2^{(3)}\left[c;1+\alpha,1+\beta,1+\gamma;\frac{-xt}{(1-t)},\frac{-yt}{(1-t)},\frac{-zt}{(1-t)}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(n!)^2(c)_n\ L_n^{(\alpha,\beta,\gamma)}(x,y,z)t^n}{(1+\alpha)_n(1+\beta)_n(1+\gamma)_n} \tag{1.23.35}$$

Laguerre Polynomials of r-Variable:

In 1997-98, the Laguerre polynomials of r-variable are defined by Khan and Shukla [165, p.163 (7.1),(7.3); see also 167,] in the following form

$$L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,\ldots,x_r)$$
$$= \frac{\prod_{j=1}^{r}(1+\alpha_j)_n}{(n!)^r}\Psi_2^{(r)}\left[-n;1+\alpha_1,1+\alpha_2,\ldots,1+\alpha_r;x_1,x_2,\ldots,x_r\right] \tag{1.23.36}$$

$$L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \binom{\alpha_1+n}{n}\cdots\binom{\alpha_r+n}{n}\Psi_2^{(r)}\left[-n;\alpha_1+1,\ldots,\alpha_r+1;x_1,\ldots,x_r\right] \tag{1.23.37}$$

where $\Psi_2^{(r)}$ denotes Humbert's confluent hypergeometric function of r variables.

Classical Jacobi Polynomial of One Variable:

The Jacobi's polynomials $P_n^{(\alpha,\beta)}(x)$ [252, p.254 (1)] are given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{array}{c} -n, 1+\alpha+\beta+n; \ \frac{1-x}{2} \\ 1+\alpha \quad ; \end{array}\right] = (-1)^n P_n^{(\beta,\alpha)}(-x) \tag{1.23.38}$$

The Laguerre's polynomials $L_n^{(\alpha)}(x)$ [252, p.200 (1)] are given by

$$\lim_{|\beta|\longrightarrow\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1\left[\begin{array}{c} -n \quad ; \ x \\ 1+\alpha; \end{array}\right] \tag{1.23.39}$$

A transformation for Jacobi polynomial [333, p. 64 (4.22.1)] is given by

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{x+3}{x-1}\right) \tag{1.23.40}$$

$$P_n^{(\alpha,\beta-n)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-n-1,\beta-n)}\left(\frac{x+3}{x-1}\right) \tag{1.23.41}$$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1+x}{2}\right)^n P_n^{(\alpha,-\alpha-\beta-2n-1)}\left(\frac{3-x}{x+1}\right) \tag{1.23.42}$$

A generating relation [326, P.90, Q.N.15] is given by

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x)t^n = (1+t)^\alpha\left[1-\frac{1}{2}(x-1)t\right]^{-\alpha-\beta-1} \tag{1.23.43}$$

and

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(x)t^n = (1-t)^\beta\left[1-\frac{1}{2}(x+1)t\right]^{-\alpha-\beta-1} \tag{1.23.44}$$

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)t^n = \left[1+\frac{1}{2}(x+1)t\right]^\alpha\left[1+\frac{1}{2}(x-1)t\right]^\beta \tag{1.23.45}$$

A generating relation [326, p.170 Q.N.19(1); see also 113, p.120 (12)] is given by

$$\sum_{n=0}^{\infty} \frac{t^n}{(1+\alpha)_n} P_n^{(\alpha,\beta-n)}(x) = exp\left[\frac{1}{2}(1+x)t\right] {}_1F_1\left[\begin{array}{c} -\beta \quad ; \ \frac{1}{2}(1-x)t \\ 1+\alpha; \end{array}\right] \tag{1.23.46}$$

Chebyshev (Tchebycheff) Polynomials of I and II Kind: [92, p.186 (24),(25)]

$$T_n(t) = \cos[n\{\cos^{-1}(t)\}] = {}_2F_1\left[\begin{array}{c} -n, \ n \ ; \ \frac{1-t}{2} \\ \frac{1}{2} \quad ; \end{array}\right] \tag{1.23.47}$$

$$U_n(t) = \frac{\sin[(n+1)\{\cos^{-1}(t)\}]}{\sqrt{(1-t^2)}} = (n+1)\ {}_2F_1\left[\begin{array}{c} -n,\ n+2\ ;\ \frac{1-t}{2} \\ \frac{3}{2}\qquad ; \end{array}\right] \tag{1.23.48}$$

Tchebycheff Polynomials of I and II Kind [252, p.301 (1,2,3,4); see also 326, p.36] is given by

$$T_n(x) = \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \tag{1.23.49}$$

$$U_n(x) = \frac{(n+1)!}{\left(\frac{3}{2}\right)_n} P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x) \tag{1.23.50}$$

$$\left(1-2xt+t^2\right)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n \tag{1.23.51}$$

$$\sum_{n=0}^{\infty} T_n(x)t^n = (1-xt)(1-2xt+t^2)^{-1} \tag{1.23.52}$$

$$\sum_{n=0}^{\infty} U_n(x)t^n = \left(1-2xt+t^2\right)^{-1} \tag{1.23.53}$$

Gegenbauer Polynomials and their Special Case:

$$\begin{aligned} C_n^{\nu}(z) &= \frac{\binom{n+2\nu-1}{n}}{\binom{n+\nu-\frac{1}{2}}{n}} P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z) \\ &= \frac{\Gamma(n+2\nu)}{n!\ \Gamma(2\nu)} \frac{n!\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(n+\nu+\frac{1}{2}\right)} P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z) \\ &= \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z) \end{aligned} \tag{1.23.54}$$

Gegenbauer Polynomial [326, p.137 (5)] is defined as

$$C_n^{\alpha}(x) = \frac{(2\alpha)_n}{n!} x^n {}_2F_1\left[\begin{array}{c} \triangle(2;-n);\ \frac{x^2-1}{x^2} \\ \alpha+\frac{1}{2}\quad ; \end{array}\right] \tag{1.23.55}$$

Gegenbauer Polynomial [252, p.280 (19)]

$$C_n^{\frac{1}{2}}(x) = P_n(x) = \frac{\left(\frac{1}{2}\right)_n (2x)^n}{n!} {}_2F_1\left[\begin{array}{c} -\frac{n}{2},\ \frac{-n+1}{2};\ \frac{1}{x^2} \\ \frac{1}{2}-n\quad ; \end{array}\right] \tag{1.23.56}$$

$$C_n^{\nu}(x) = \frac{(\nu)_n (2x)^n}{n!} {}_2F_1\left[\begin{array}{c} -\frac{n}{2},\ \frac{-n+1}{2};\ \frac{1}{x^2} \\ 1-\nu-n; \end{array}\right] \tag{1.23.57}$$

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n \tag{1.23.58}$$

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.23.59}$$

Gegenbauer or Ultraspherical polynomials [333, p.81 (4.7.1)]

$$P_n^{(\nu)}(x) = \left[\frac{(2\nu)_n}{(\nu+\frac{1}{2})_n}\right] P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x) = C_n^{\nu}(x); \qquad \nu > -\frac{1}{2} \tag{1.23.60}$$

$$P_n^{(\alpha,\alpha)}(x) = \binom{\alpha+n}{n}\binom{2\alpha+n}{n}^{-1} C_n^{\alpha+\frac{1}{2}}(x) \tag{1.23.61}$$

$$P_n^{(0,0)}(x) = C_n^{\frac{1}{2}}(x) = P_n^{(\frac{1}{2})}(x) = P_n(x) \tag{1.23.62}$$

$$U_n(x) = C_n^1(x) = P_n^{(1)}(x) \tag{1.23.63}$$

A generating relation for Gegenbauer polynomial [252, p.278 (7)] is given by

$$e^{xt}{}_0F_1\left[\begin{matrix} - & ; \\ \nu+\frac{1}{2}; \end{matrix} \frac{t^2(x^2-1)}{4}\right] = {}_0F_0\left[\begin{matrix} -; \\ -; \end{matrix} xt\right]{}_0F_1\left[\begin{matrix} - & ; \\ \nu+\frac{1}{2}; \end{matrix} \frac{t^2(x^2-1)}{4}\right] = \sum_{n=0}^{\infty} \frac{C_n^{\nu}(x)t^n}{(2\nu)_n} \tag{1.23.64}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} C_n^{\alpha}(x)t^n = F_1\left[\lambda;\alpha,\alpha;\mu;(x+\sqrt{x^2-1})t,(x-\sqrt{x^2-1})t\right] \tag{1.23.65}$$

Rice and Khandekar Polynomials of One Variable:

In the endeavor of attempting to unify several results in the theory of polynomials, also in hypergeometric functions of two or more variables, Khandekar [171, p.158 (2.3)] defined the generalized Rice polynomials of one variable in the following form:-

$$H_n^{(\alpha,\beta)}[\xi,p,\nu] = \frac{(1+\alpha)_n}{n!}\,{}_3F_2\left[\begin{matrix} -n,\ 1+\alpha+\beta+n,\ \xi & ; \\ & & \nu \\ 1+\alpha,\ p & ; \end{matrix}\right] \tag{1.23.66}$$

where $n = 0, 1, 2, 3, \ldots.,$ and $p \neq -n-1, -n-2, -n-3, \ldots.$

If $\alpha = \beta = 0$ in (1.23.66), we get Rice polynomials [253, p.108]

$$H_n[\xi,p,\nu] = H_n^{(0,0)}[\xi,p,\nu] \tag{1.23.67}$$

If $p = \xi$ and $\nu = \frac{1}{2}(1-x)$, (1.23.66) becomes:-

$$H_n^{(\alpha,\beta)}\left[\xi,\xi,\frac{1}{2}(1-x)\right] = P_n^{(\alpha,\beta)}(x) \tag{1.23.68}$$

or

$$H_n^{(\alpha,\beta)}[\xi,\xi,x] = P_n^{(\alpha,\beta)}(1-2x) \tag{1.23.69}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials.

Multivariable Generalized Rice Polynomials of Qureshi-Kabra-Quraishi: Motivated by the works on different types of Hypergeometric polynomials of one and more variables, we define a generalized Rice's polynomials of r variables in the following form:-

$$H_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}\Big[(b_{B_1}^{(1)});(d_{D_1}^{(1)}):\ldots:(b_{B_r}^{(r)});(d_{D_r}^{(r)}):x_1,\ldots,x_r\Big]=\frac{(1+\alpha_1)_n\ldots(1+\alpha_r)_n}{(n!)^r}\times$$

$$\times F_{0:1+D_1;\ldots;1+D_r}^{1:1+B_1;\ldots;1+B_r}\left[\begin{array}{ccccc} -n: 1+\alpha_1+\beta_1+n,(b_{B_1}^{(1)});\ldots;1+\alpha_r+\beta_r+n,(b_{B_r}^{(r)}); & \\ -\ :\quad 1+\alpha_1,(d_{D_1}^{(1)})\quad ;\ldots;\quad 1+\alpha_r,(d_{D_r}^{(r)})\quad ; & \end{array} x_1,\ldots,x_r\right] \tag{1.23.70}$$

where $F_{0:1+D_1;\ldots;1+D_r}^{1:1+B_1;\ldots;1+B_r}[\ldots]$ is multivariable Kampé de Fériet function and $(b_{B_r}^{(r)})$ denotes B_r parameters given by $b_1^{(r)},b_2^{(r)},\ldots,b_{B_r}^{(r)}$ with similar interpretation for others.

On specializing the parameters and arguments, the Qureshi-Kabra-Quraishi polynomials reduce to generalized Sister Celine's polynomials of Shah[263, p.80 (2.2)], Fasenmyer Sister Celine's polynomials [111, 112], multivariable Sister Celine's polynomials of Shrivastava [273, 286-288], Bateman's polynomials $Z_n(x)$ and $F_n(z)$ [20-22], Pasternack's polynomials $F_n^{(m)}(z)$ [215], generalized Rice's polynomials of Khandekar $H_n^{(\alpha,\beta)}[\xi,p,\nu]$ [171, p.158 ; see also 79, 191, 304, 309], Rice's polynomials $H_n[\xi,p,\nu]$ [253, p.108], Classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [252, pp. 254, 255], Classical Legendre polynomials $P_n(x)$ [252, pp. 166, 167, 183, 185] and classical Ultraspherical (Gegenbauer) polynomials $C_n^{\nu}(x)$ [252, pp. 279, 280], Classical Chebyshev polynomials of I kind $T_n(x)$ and II kind $U_n(x)$ [252, pp. 301, 302], Cohen polynomials [58, 59], Extended Jacobi polynomials of Fujiwara $F_n^{(\alpha,\beta)}(x;a,b,c)$ [115], Khan polynomials [154, 155, 157], Khan-Shukla polynomials [168], multivariable Jacobi polynomials of Shrivastava's [271, 274, 281, 283, 284] and others.

Classical Extended Jacobi Polynomial of Fujiwara:[115, p.136]

The polynomials $F_n^{(\alpha,\beta)}(x)$ are equivalent to (and not a generalization of) the classical extended Jacobi polynomials.

$$F_n^{(\alpha,\beta)}(x;a,b,c) = c^n(a-b)^n P_n^{(\alpha.\beta)}\left(2\left\{\frac{x-a}{a-b}\right\}+1\right) = c^n(b-a)^n P_n^{(\alpha,\beta)}(y), \tag{1.23.71}$$

where

$$y = \frac{2x}{(b-a)} - \frac{(a+b)}{(b-a)}$$

An identity [333, p.58 (4.1.2)] is given by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{c^n(a-b)^n} F_n^{(\alpha.\beta)}\left(\frac{1}{2}\left\{a+b+(a-b)x\right\}; a,b,c\right) \tag{1.23.72}$$

The well-known relationship [228, p. 389 (2.7)] is given by

$$F_n^{(\alpha,\beta)}(a+b-x;a,b,c) = (-1)^n F_n^{(\beta,\alpha)}(x;a,b,c) \tag{1.23.73}$$

The generating function [228, p. 391 (2.15)] is given by

$$\begin{aligned} F_n^{(\alpha,\beta)}(x;a,b,c) &= \left(-\frac{x-a}{a-b}\right)^n F_n^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{ax-2ab+b^2}{x-a}; a,b,c\right) \\ &= \left(\frac{x-b}{a-b}\right)^n F_n^{(\alpha,-\alpha-\beta-2n-1)}\left(\frac{bx-2ab+a^2}{x-b}; a,b,c\right) \end{aligned} \tag{1.23.74}$$

The Jacobi Polynomials of Two Variables:

$$\begin{aligned} P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(x,y) &= \frac{\Gamma(1+\alpha_1+n)\Gamma(1+\alpha_2+n)}{(n)!}\times \\ &\times \sum_{j=0}^{n} \frac{(-1)^j(1+\alpha_2+\beta_2+n)_j}{(j)!\Gamma(1+\alpha_1+n-j)\Gamma(1+\alpha_2+j)}\left(\frac{1-y}{2}\right)^j P_{(n-j)}^{(\alpha_1,\beta_1)}(x) \end{aligned} \tag{1.23.75}$$

$$\begin{aligned} P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(x,y) &= \frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}\times \\ &\times F_2\left[-n; 1+\alpha_2+\beta_2+n, 1+\alpha_1+\beta_1+n; 1+\alpha_2, 1+\alpha_1; \frac{1-y}{2}, \frac{1-x}{2}\right] \end{aligned} \tag{1.23.76}$$

Jacobi Polynomials of Three Variables: [271, p.63 (8)], is defined in the following form

$$P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2;\alpha_3,\beta_3)}(x_1,x_2,x_3) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}{(n!)^3}\times$$

$$\times F^{(3)}\left[\begin{array}{l} -n :: -;-;- : 1+\alpha_3+\beta_3+n; 1+\alpha_2+\beta_2+n; 1+\alpha_1+\beta_1+n; \\ - :: -;-;- : \quad 1+\alpha_3 \quad ; \quad 1+\alpha_2 \quad ; \quad 1+\alpha_1 \quad ; \end{array} \frac{1-x_3}{2}\frac{1-x_2}{2},\frac{1-x_1}{2}\right] \quad (1.23.77)$$

$$P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,x_2,\ldots,x_r) = \frac{(1+\alpha_1)_n\ldots(1+\alpha_r)_n}{(n!)^r}\times$$

$$\times F^{1:1;\ldots;1}_{0:1;\ldots;1}\left[\begin{array}{l} -n : 1+\alpha_r+\beta_r+n;\ldots;1+\alpha_1+\beta_1+n; \\ - : \quad 1+\alpha_r \quad ;\ldots; \quad 1+\alpha_1 \quad ; \end{array} \frac{1-x_r}{2},\ldots,\frac{1-x_1}{2}\right] \quad (1.23.78)$$

The generalized Jacobi polynomials of r variables [271, p.65 (15)] are defined by

$$P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) = \frac{(1+\alpha_1)_n\ldots(1+\alpha_r)_n}{(n!)^r}\times$$

$$\times F^{1:1;\ldots;1}_{0:1;\ldots;1}\left[\begin{array}{l} -n : 1+\alpha_1+\beta_1+n;\ldots;1+\alpha_r+\beta_r+n; \\ - : \quad 1+\alpha_1 \quad ;\ldots; \quad 1+\alpha_r \quad ; \end{array} \frac{1-x_1}{2},\ldots,\frac{1-x_r}{2}\right], \quad (1.23.79)$$

where $F^{1:1;\ldots;1}_{0:1;\ldots;1}[\ldots]$ is multivariable Kampé de Fériet function.

$$P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) = \binom{\alpha_1+n}{n}\ldots\binom{\alpha_r+n}{n}\times$$

$$\times F_A^{(r)}\left[-n : \alpha_1+\beta_1+n+1,\ldots,\alpha_r+\beta_r+n+1; \alpha_1+1,\ldots,\alpha_r+1; \frac{1-x_1}{2},\ldots,\frac{1-x_r}{2}\right] \quad (1.23.80)$$

$$\lim_{min(|\beta_1|,\ldots,|\beta_r|)\longrightarrow\infty}\left\{P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}\left(1-\frac{2x_1}{\beta_1},\ldots,1-\frac{2x_r}{\beta_r}\right)\right\}$$

$$= L_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) \quad (1.23.81)$$

Properties of the multivariable Jacobi polynomials [41, p.1552 (4.11)] is given by

$$P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) = \prod_{j=1}^{r}\left\{\binom{\alpha_j+n}{n}\right\}\times$$

$$\times\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha_r+\beta_r+n+1)_k}{(\alpha_r+1)_k}\left(\frac{x_r-1}{2}\right)^k\prod_{j=1}^{r-1}\left\{\binom{\alpha_j+n-k}{n-k}^{-1}\right\}\times$$

$$\times P_{n-k}^{(\alpha_1,\beta_1+k;\ldots;\alpha_{r-1},\beta_{r-1}+k)}(x_1,\ldots,x_{r-1}) \quad (1.23.82)$$

which is the corrected version of a known formula [271, p.66 (17)].

A relationship between Jacobi polynomials [41, p.1552 (4.12)] is given by

$$P_n^{(\alpha_1,\beta_1;\ldots;\alpha_r,\beta_r)}(x_1,\ldots,x_r) = \left(\frac{1+x_1}{2}\right)^n \times$$

$$\times\, P_n^{(\alpha_1,-\alpha_1-\beta_1-2n-1;\alpha_2,\beta_2;\ldots;\alpha_r,\beta_r)}\left(\frac{3-x_1}{1+x_1},\frac{x_1+2x_2-1}{1+x_1},\ldots,\frac{x_1+2x_r-1}{1+x_1}\right) \tag{1.23.83}$$

Krall and Frink Bessel-Polynomials:

Krall and Frink Bessel's polynomial is defined as

$$y_n(x,\alpha-n,\beta) = n!\,\left(\frac{-x}{\beta}\right)^n\; L_n^{(1-\alpha-n)}\left(\frac{\beta}{x}\right) \tag{1.23.84}$$

Krall and Frink [326, p.170 Q.19(ii)] generating relation is given by

$$\sum_{n=0}^{\infty} y_n(x,a-n,b)\frac{t^n}{n!} = \left[1-\frac{xt}{b}\right]^{1-a} e^t \tag{1.23.85}$$

Rainville's Polynomials:[252]

$$g_n(x,a,b) = (2x)^n\,{}_3F_1\left[\begin{matrix}\frac{-n}{2},\frac{-n+1}{2},1+a; & -\frac{1}{x^2}\\ 1+b \quad ; & \end{matrix}\right] \tag{1.23.86}$$

$$\sum_{n=0}^{\infty} g_n(x,a,b)\frac{t^n}{n!} = e^{2xt}\,{}_1F_1\left[\begin{matrix}1+a; & -t^2\\ 1+b; & \end{matrix}\right] \tag{1.23.87}$$

When $b=a$, we get

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{1.23.88}$$

Two generating functions [252, p.290 (7),(8)] are given by

$$(1+t)^z(1-t)^{-z} = 1+\sum_{n=1}^{\infty} g_n(z)t^n \tag{1.23.89}$$

where

$$g_n(z) = 2z\,{}_2F_1\left(1-n,1-z;2;2\right) \tag{1.23.90}$$

$$2ze^t\,{}_1F_1\left(1-z;2;-2t\right) = \sum_{n=0}^{\infty}\frac{g_{n+1}(z)t^n}{n!}. \tag{1.23.91}$$

A polynomial and generating function [252, p.298 (1),(4)] is given by

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\,{}_1F_1(-n;a+n;x), \tag{1.23.92}$$

which are related to the proper simple Laguerre polynomial

$$L_n(x) = {}_1F_1(-n; 1; x) \tag{1.23.93}$$

$$e^{2t}{}_0F_1\left(-; \frac{1}{2} + \frac{a}{2}; t^2 - xt\right) = \sum_{n=0}^{\infty} \frac{R_n(a, x)t^n}{\left(\frac{1}{2} + \frac{a}{2}\right)_n} \tag{1.23.94}$$

Generating relations [252, P.302 Q.3,Q.4] are defined as

$$\Psi_n(c, x, y) = \frac{(-1)^n \left(\frac{1}{2} + \frac{x}{2}\right)_n}{(c)_n} {}_3F_2 \left[\begin{array}{ll} -n, \frac{1}{2} - \frac{x}{2}, 1 - c - n; & \\ c, \quad \frac{1}{2} - \frac{x}{2} - n \quad ; & y \end{array} \right] \tag{1.23.95}$$

$${}_1F_1\left(\frac{1}{2} - \frac{x}{2}; c; yt\right) {}_1F_1\left(\frac{1}{2} + \frac{x}{2}; c; -t\right) = \sum_{n=0}^{\infty} \frac{\Psi_n(c, x, y)t^n}{n!} \tag{1.23.96}$$

$$\Phi_n(x) = \frac{x^n}{n!} {}_2F_0\left(-n, x; -; -\frac{1}{x}\right) \tag{1.23.97}$$

$$(1 - t)^{-x} e^{xt} = \sum_{n=0}^{\infty} \Phi_n(x) t^n \tag{1.23.98}$$

Bateman's generating relation [252, p.290 (6)] is given by

$${}_1F_1\left(\begin{array}{c} \frac{1}{2} - \frac{z}{2}; \\ 1; \end{array} t \right) {}_1F_1\left(\begin{array}{c} \frac{1}{2} + \frac{z}{2}; \\ 1; \end{array} -t \right) = \sum_{n=0}^{\infty} \frac{F_n(z)t^n}{n!} \tag{1.23.99}$$

Devisme's Polynomials: [95, p.268 (19.11.4)]

Explicit (but Complicate) expression for the polynomials were given by Devisme (1932,1933)

$$e^{xt - yt^2 + \frac{t^3}{3}} = \sum_{n=0}^{\infty} U_n(x, y) t^n \tag{1.23.100}$$

$$\sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!} = exp(2xt - yt^2 + zt^3) \tag{1.23.101}$$

Devisme (1932, 1933) [95, p.247 (19.6.14); see also 333]

$$(1 - 3xt + 3yt^2 - t^3)^{-\nu} = \sum_{n=0}^{\infty} H_n^{\nu}(x, y) t^n \tag{1.23.102}$$

Devisme (1936)

$$[1 - x^m + (x - t)^m]^{-\nu} = \sum_{n=0}^{\infty} {}_mC_n^{\nu}(x) t^n \tag{1.23.103}$$

Gould-Hopper Polynomials:[326, p.76 cf. (1.9) (6)]

$$\sum_{n=0}^{\infty} g_n^m(x,h)\frac{t^n}{n!} = exp(xt + ht^m) \tag{1.23.104}$$

where m is positive integer $m \geq 2$

Replace x by $2x$, $h = -1$, $m = 2$ we get a classical Hermite polynomials $H_n(x)$

$$H_n(x) = g_n^2(2x, -1) \tag{1.23.105}$$

Special case $m = 2, h = -1$ Gould-Hopper reduces to Dattoli polynomial.

$$g_n^m(x,h) = x^n {}_mF_0\left[\begin{array}{c}\Delta(m;-n);\\ -\quad ;\end{array} h\left(\frac{-m}{x}\right)^m\right] \tag{1.23.106}$$

Humbert Polynomial and It's Special Case:

Generalized Humbert Polynomials [326, P.86 (26),(27)]

$$(1 - mxt + t^m)^{-\nu} = \sum_{n=0}^{\infty} h_{n,m}^{\nu}(x)t^n \tag{1.23.107}$$

where m is a positive integer.

A generating relation of Pincherle is given by

$$(1 - 3xt + t^3)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} g_n(x)t^n \tag{1.23.108}$$

A generating relation of Humbert [132] is given by

$$(1 - 3xt + t^3)^{-\nu} = \sum_{n=0}^{\infty} h_{n,3}^{\nu}(x)t^n \tag{1.23.109}$$

A polynomial of Lahiri [177, p.118 (3.2)] is given by

$$H_{n,m,\nu}(x) = \nu^n g_n^m(x, -1) = g_n^m(\nu x, -1) \tag{1.23.110}$$

A polynomial of Gupta and Jain [121] is given by

$$H_{n,m,\nu}(x) = \nu^n H_{n,m}(x, -1) = H_{n,m}(\nu x, -1) \tag{1.23.111}$$

A polynomial of Srivastava [314] is given by

$$H_{n,m,\nu}(x) = (\nu x)^n {}_mF_0\left[\begin{array}{c}\triangle(m;-n);\\ -\ ;\end{array} -\left(\frac{-m}{\nu x}\right)^m\right] \tag{1.23.112}$$

The explicit form of the polynomial $P_{n,m}^{\lambda}(x)$ is [Equation 7.1.3, pp. 135]

$$\sum_{n=0}^{\infty} P_{n,m}^{\lambda}(x)t^n = (1 - mxt + t^m)^{-\lambda} \tag{1.23.113}$$

where

$$P_{n,m}^{\lambda}(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k (\lambda)_{n-(m-1)k} (2x)^{n-mk}}{k!(n - m^k)!} \tag{1.23.114}$$

Horadam polynomials [127, p.296 (2.6)]

$$P_{n,1}^{\lambda}(x) = \frac{(\lambda)_n}{n!}(2x - 1)^n \tag{1.23.115}$$

Gegenbauer polynomials

$$P_{n,2}^{\lambda}(x) = C_n^{\lambda}(x) \tag{1.23.116}$$

Horadam-Pethe polynomials [128]

$$P_{n,3}^{\lambda}(x) = P_{n+1}^{\lambda}(x) \tag{1.23.117}$$

Gould Generalized Humbert Polynomial and Their Special Case:

The object of this section is to present some novel expansions, in particular a pair of inverse series relations, which occur in the study of a general class of polynomials defined as follows. By a generalized Humbert polynomial $P_n(m, x, y, p, C)$ we mean the coefficient determined by the series expansion of $H = H(t; m, x, y, p, C) = (C - mxt + yt^m)^p$,

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n \tag{1.23.118}$$

where $m \geq 1$ is an integer and the other parameters are unrestricted in general. That P_n is a polynomial in x will be made evident later by means of explicit formulas.

This definition includes many well-known and not so well-known special cases. We tabulate the main cases, each name being followed by a year:

$$P_n(2, q, -1, -\frac{1}{2}, p^2) = f_n(p, q), \qquad Louville(1722)$$

$$P_n(2, x, 1, -\frac{1}{2}, 1) = X_n = P_n(x) \qquad Legendre(1784)$$

$$P_n(2, x, 1, -1, 1) = U_n(x), \qquad Tchebycheff(1859)$$

$$P_n(2,x,1,-\nu,1) = C_n^{\nu}(x), \qquad Gegenbauer(1874)$$
$$P_n(3,x,1,-\frac{1}{2},1) = \text{“}P_n(x)\text{”}, \qquad Pincherle(1890)$$
$$P_n(m,x,1,-\nu,1) = \prod_{n,m}^{\nu}(x), \qquad Humbert(1921)$$
$$P_n(m,x,1,-\frac{1}{m},1) = P_n(m,x), \qquad Kinney(1963)$$

Gould generalized Humbert polynomials [326, p.86 cf. 26, 1.9(13)]

$$(c - mxt + yt^m)^q = \sum_{n=0}^{\infty} P_n(m,x,y,q,c)t^n \tag{1.23.119}$$

where m is a positive integer.

The polynomial $P_n(m,x,y,q,c)$ [117, p. 697]

$P_n(m,x,y,q,c) \longrightarrow C_n^q(x), P_n(x), U_n(x), h_{n,m}^{-q}(x)$

The polynomial $P_n(m,x,y,q,c)$ [117, p. 699] defined as

$$P_n(m,x,y,q,c) = \sum_{k=0}^{[\frac{n}{m}]} \binom{q}{k}\binom{q-k}{n-mk} c^{q-n+(m-1)k} y^k (-mx)^{n-mk} \tag{1.23.120}$$

where m is positive integer and other parameters are unrestricted in general.

Bedient Polynomials:

Bedient polynomials [252, p.(297) (1, 2, 3, 5)]

$$G_n(\alpha,\beta;x) = \frac{(\alpha)_n(\beta)_n}{n!(\alpha+\beta)_n}(2x)^n {}_3F_2\left[\begin{array}{ll} \triangle(2;-n), 1-\alpha-\beta-n; & \frac{1}{x^2} \\ 1-\alpha-n, \qquad 1-\beta-n; & \end{array}\right] \tag{1.23.121}$$

Bedient's polynomials [326, P.186 Q.48]

$$R_n(\beta,\gamma;x) = \frac{(\beta)_n}{n!}(2x)^n {}_3F_2\left[\begin{array}{ll} \triangle(2;-n), \gamma-\beta; & \frac{1}{x^2} \\ \gamma, \qquad 1-\beta-n; & \end{array}\right] \tag{1.23.122}$$

$$\lim_{\alpha\longrightarrow\infty} G_n(\alpha,\beta;x) = C_n^{\beta}(x) \tag{1.23.123}$$

$$\lim_{\beta\longrightarrow\infty} G_n(\alpha,\beta;x) = C_n^{\alpha}(x) \tag{1.23.124}$$

$$\lim_{\gamma\longrightarrow\infty} R_n(\beta,\gamma;x) = C_n^{\beta}(x) \tag{1.23.125}$$

Bedient polynomials [326, p.186]

$$\sum_{n=0}^{\infty} \frac{[(a_A)]_n(\alpha+\beta)_n}{[(b_B)]_n} G_n(\alpha,\beta;\gamma)t^n = F_{B:0;0}^{A:2;2}\left[\begin{array}{ll} (a_A) : \alpha,\beta;\alpha;\beta; & \mu t, \nu t \\ (b_B) : \quad -; \quad -; & \end{array}\right] \tag{1.23.126}$$

$$\sum_{n=0}^{\infty}\frac{[(a_A)]_n}{[(b_B)]_n(\gamma)_n}R_n(\beta;\gamma;x)t^n = F^{A:1;1}_{B:1;1}\left[\begin{array}{c}(a_A):\beta;\beta;\\(b_B):\gamma;\gamma;\end{array}\mu t,\nu t\right] \tag{1.23.127}$$

Bedient polynomials [252, p.298 (6,7)]

$$\sum_{n=0}^{\infty}G_n(\alpha,\beta;x)t^n = {}_2F_1\left(\alpha,\beta;\alpha+\beta;2xt-t^2\right) \tag{1.23.128}$$

$$\sum_{n=0}^{\infty}(\alpha+\beta)_nG_n(\alpha,\beta;x)t^n \cong {}_2F_0\left[\alpha,\beta;-;t(x-\sqrt{x^2-1})\right]{}_2F_0\left[\alpha,\beta;-;t(x+\sqrt{x^2-1})\right] \tag{1.23.129}$$

$$\sum_{n=0}^{\infty}\frac{R_n(\beta,\gamma;x)t^n}{(\gamma)_n} = {}_1F_1\left[\beta;\gamma;t(x-\sqrt{x^2-1})\right]{}_1F_1\left[\beta;\gamma;t(x+\sqrt{x^2-1})\right] \tag{1.23.130}$$

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(\gamma)_n}G_n(\alpha,\beta;x)t^n = F_3\left[\alpha,\alpha,\beta,\beta;\gamma;\mu t,\nu t\right] \tag{1.23.131}$$

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(\gamma)_n}G_n(\alpha,\beta;x)t^n = F^{0:2;2}_{1:0;0}\left[\begin{array}{c}-:\alpha,\beta;\alpha,\beta;\\\gamma:\quad -;\quad -;\end{array}\mu t,\nu t\right] \tag{1.23.132}$$

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}R_n(\beta,\gamma;x)t^n = F_2\left[\alpha,\beta,\beta;\gamma,\gamma;\mu t,\nu t\right] \tag{1.23.133}$$

where $\mu = x-\sqrt{x^2-1}$ and $\nu = x+\sqrt{x^2-1}$.

Multivariable Lagrange Polynomials of Chan, Chen and Srivastava:

Lagrange's polynomial of three variables [166]

$$\sum_{n=0}^{\infty}g_n^{(\alpha,\beta,\gamma)}(x,y,z)\frac{t^n}{n!} = (1-xt)^{-\alpha}(1-yt)^{-\beta}(1-zt)^{-\gamma} \tag{1.23.134}$$

Lagrange's polynomial of r variables [42, p.140 (6)]

$$(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2}\dots(1-x_rt)^{-\alpha_r} = \sum_{n=0}^{\infty}g_n^{(\alpha_1,\alpha_2,\dots,\alpha_r)}(x_1,x_2,\dots,x_r)t^n \tag{1.23.135}$$

$$\alpha_1,\alpha_2,\dots,\alpha_r\in\mathbb{C};\qquad |t|<min\{|x_1|^{-1},|x_2|^{-1},\dots,|x_r|^{-1}\}$$

Addition formula [42, p.147 (35)]

$$g_n^{(\alpha_1+\beta_1,\dots,\alpha_r+\beta_r)}(x_1,\dots,x_r) = \sum_{k=0}^{n}g_{n-k}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)g_k^{(\beta_1,\dots,\beta_r)}(x_1,\dots,x_r) \tag{1.23.136}$$

The relationship can be used in order to reduce numerous properties and characteristics of the (two-variable) Lagrange's polynomials from those of the classical Jacobi polynomials [326, p.442 (8.5.17)]

$$g_n^{(\alpha,\beta)}(x,y) = (y-x)^n P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right) \tag{1.23.137}$$

A transformation formula for Jacobi polynomial [326, p.441 (8.5.16)] is given by

$$P_n^{(\alpha-n,\beta-n)}(x) = g_n^{(-\alpha,-\beta)}\left(-\frac{x+1}{2}, -\frac{x-1}{2}\right) \tag{1.23.138}$$

A transformation formula [326, p.452 Q.25; see also 325, p.318 (103)] is given by

$$g_n^{(\alpha,\beta)}(x,y) = y^n P_n^{(\alpha+\beta-1,-\beta-n)}\left(\frac{2x-y}{y}\right) \tag{1.23.139}$$

Put $y = \frac{2x}{x+1}$ in above equation, we get

$$P_n^{(\alpha+\beta-1,-\beta-n)}(x) = \left(\frac{x+1}{2x}\right)^n g_n^{(\alpha,\beta)}\left(x, \frac{2x}{x+1}\right) \tag{1.23.140}$$

Two-variable Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ [54, p.254 (3.5)]

$$g_n^{(\alpha,\beta-n)}(x,y) = g_n^{(\alpha,-\alpha-\beta+1)}(x-y,-y) \tag{1.23.141}$$

Property of Lagrange polynomials [54, p.255 (3.6), (3.7)]

$$g_n^{(\alpha,\beta)}(x,y) = g_n^{(\beta,\alpha)}(y,x) \tag{1.23.142}$$

The relationships [54, p.255 (3.9)] is given by

$$\begin{aligned} g_n^{(\alpha,\beta-n)}(x,y) = g_n^{(\alpha,-\alpha-\beta+1)}(x-y,-y) &= (x)^n P_n^{(-\alpha-n,\alpha+\beta-n-1)}\left(\frac{x-2y}{x}\right) \\ &= (x)^n P_n^{(\alpha+\beta-n-1,-\alpha-n)}\left(\frac{2y-x}{x}\right) \end{aligned} \tag{1.23.143}$$

$$P_n^{(\alpha+\beta-n-1,-\alpha-n)}(x) = x^{-n} g_n^{(\alpha,\beta-n)}\left(x, \frac{x(x+1)}{2}\right) \tag{1.23.144}$$

Lagrange-Hermite Polynomial of r Variables of Khan-Shukla: [166]

Lagrange-Hermite polynomial of three variables

$$\sum_{n=0}^{\infty} h_n^{(\alpha,\beta,\gamma)}(x,y,z)\frac{t^n}{n!} = (1-xt)^{-\alpha}(1-yt^2)^{-\beta}(1-zt^3)^{-\gamma} \tag{1.23.145}$$

$$\sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(x_1,x_2,\ldots,x_r)\frac{t^n}{n!} = (1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\ldots,(1-x_rt^r)^{-\alpha_r} \tag{1.23.146}$$

Chapter 2

Some Linear, Bilinear, Bilateral Generating Relations and Summation Formulae

2.1 Introduction

Linear Generating Functions:

Consider a two-variable function $F(x,t)$ which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in t such that

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of t. Then the expansion of $F(x,t)$ is said to have generated the set $f_n(x)$ and $F(x,t)$ is called a linear generating function (or, simply, a generating function) for the set $f_n(x)$.

The definition may be extended slightly to include a generating function of the type

$$G(x,t) = \sum_{n=0}^{\infty} c_n g_n(x)t^n \tag{2.1.1}$$

where the sequence $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of x and t. A set of functions may have more than one generating function. However, if $G(x,t) = \sum_{n=0}^{\infty} h_n(x)t^n$ then $G(x,t)$ is the unique generator for the set $h_n(x)$ as the coefficient set.

Now we, extend our definition of a generating relation to include functions which possess Laurent series expansion. Thus, if the set $f_n(x)$ is defined for $n = 0, \pm 1, \pm 2, \ldots$, the definition (2.1.1) may be extended in terms of Laurent series expansion

$$F^*(x,t) = \sum_{n=-\infty}^{\infty} \gamma_n f_n(x)t^n$$

where the sequence $\{\gamma_n(x)\}_{n=0}^{\infty}$ is independent of x and t.

Bilinear Generating Functions:

If a three-variable function $F(x,y,t)$ possesses a formal power series expansion in t such that

$$F(x,y,t) = \sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n,$$

where the sequence $\{\gamma_n\}$ is independent of x, y and t, then $F(x, y, t)$ is called a Bilinear generating function for the set $\{f_n(x)\}$.

More generally, if $\mathcal{F}(x, y, t)$ can be expanded in powers of t in the form

$$\mathcal{F}(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n,$$

where $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal, we shall still call $\mathcal{F}(x, y, t)$ a bilinear generating function for the set $\{f_n(x)\}$.

Bilateral Generating Functions:

Suppose that a three-variable function $H(x, y, t)$ has a formal power series expansion in t such that

$$H(x, y, t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n,$$

where the sequence $\{h_n\}_\infty$ is independent of x, y and t, and the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different. Then $H(x, y, t)$ is called a bilateral generating function for the set $\{f_n(x)\}$ or $\{g_n(x)\}$.

The above definition of a bilateral generating function, used earlier by Rainville [252, p.170] and McBride [199, p.19], may be extended to include bilateral generating functions of the type

$$\mathcal{H}(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n,$$

where the sequence $\{\gamma_n\}_\infty$ is independent of x, y and t, the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different, and $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal.

Extended Linear Generating Functions:

An extended linear generating function is a generating function of the type

$$F(m; x, t) = \sum_{n=0}^{\infty} \lambda_{m,n} f_{m+n}(x) t^n$$

where m is a fixed integer ≥ 0.

In this chapter, we find some summation formulas and generating relation associated with Gauss hypergeometric polynomial, are obtained by means of series rearrangement techniques and some algebraic properties of Pochhammer's symbol.

In 2007, Alhaidary [4, p.41 (19)] solved the integral $\int_0^\infty x^\nu exp\left(-\frac{x}{2}\right) L_n^{2\nu}(x) J_\nu(\mu x)\mathrm{d}x$, by means of orthogonality property of polynomials, tridiagonal matrix representation, three-term recursion relation, good knowledge of differential equation and properties of Bessel function etc.

In 2008, Alassar, Mavromatis and Sofianos [2, p.265 (7),(11)] solved the same integral $\int_0^\infty x^\nu exp\left(-\frac{x}{2}\right) L_n^{2\nu}(x) J_\nu(\mu x)\mathrm{d}x$, by means of good knowledge of hypergeometric integrals involving special functions.

Solving above same integral by two different methods and equating them, Alassar, Mavromatis and Sofianos [2, p.266 (13)] proved the following summation formula

$$\sum_{k=0}^{n} \frac{(-2)^k n!}{(1+z)^k k!(n-k)!} {}_2F_1\left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; -z\right] = {}_2F_1\left[2\nu+n+1, -n; \nu+1; \frac{1}{z+1}\right] \tag{2.1.2}$$

The object of this chapter is to apply the concept of fractional derivatives to obtain generating functions. In the process we build up a fractional derivative operator which play the role of augmenting parameters in the hypergeometric functions involved. We then employ this operator in identities involving infinite series and this exercise culminates in linear as well as bilinear generating functions for a variety of special functions.

We also obtain a Bilinear generating relation for the restricted Jacobi polynomials, using the fractional derivative technique. By the process of confluence, a number of interesting Linear, Bilinear and Bilateral generating relations for the restricted Jacobi and Laguerre polynomials, are derived as special cases. Known generating relations of Khan [153, 156] are also deduced.

Furthering the findings we establish a generating relation for the product of two restricted Jacobi polynomials, using the fractional derivative operator (1.2.26). A number of interesting generating formulae for Jacobi and Laguerre polynomials are obtained as special cases.

2.2 Linear Generating Relation and Summations

When $\nu \neq -1, -2, \ldots$ then without any loss of convergence, we have the following results

$$\sum_{k=0}^{n} \binom{n}{k} {}_2F_1\left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; bz\right] \frac{t^k}{(1+y)^k}$$

$$= \sum_{k=0}^{n} \frac{(-t)^k(-n)_k}{(1+y)^k k!} {}_2F_1\left[\frac{-n+k}{2}, \frac{-n+k+1}{2}; 1+\nu; \frac{t^2 bz}{(1+y)^2}\right] \qquad (2.2.1)$$

The right hand side of (2.2.1) is generating function of ${}_2F_1\left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; bz\right]$. By suitable adjustment of parameters and variables in generating relation (2.2.1), we obtain following summation formulas,

$$\sum_{k=0}^{n} \frac{(-2)^k n!}{(1+z)^k k!(n-k)!} {}_2F_1\left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; z\right]$$

$$= \frac{1}{(1+z)^n}\left[\frac{2z}{(z-1)-\sqrt{(z^2-6z+1)}}\right]^n {}_2F_1\left[-n, -\nu-n; 1+\nu; w_1\right] \qquad (2.2.2)$$

$$= \frac{1}{(1+z)^n}\left[\frac{2z}{(z-1)+\sqrt{(z^2-6z+1)}}\right]^n {}_2F_1\left[-n, -\nu-n; 1+\nu; w_2\right] \qquad (2.2.3)$$

$$\sum_{k=0}^{n} \frac{(2)^k(-n)_k}{(1+z)^k k!} {}_2F_1\left[\frac{-n+k}{2}, \frac{-n+k+1}{2}; 1+\nu; \frac{-4z}{(1+z)^2}\right]$$

$$= {}_2F_1\left[2\nu+n+1, -n; 1+\nu; \frac{1}{(1+z)}\right] \qquad (2.2.4)$$

where

$$2zw_1 = (z^2+1-4z) + (1-z)\sqrt{(z^2-6z+1)} \qquad (2.2.5)$$

$$2zw_2 = (z^2+1-4z) - (1-z)\sqrt{(z^2-6z+1)} \qquad (2.2.6)$$

2.3 Linear Generating Relation using Series Rearrangement Technique

Suppose left hand side of (2.2.1) is denoted by Ω, then

$$\Omega = \sum_{k=0}^{n} \frac{n! t^k}{(1+y)^k k!(n-k)!} \sum_{r=0}^{[\frac{k}{2}]} \frac{\left(\frac{-k}{2}\right)_r \left(\frac{-k+1}{2}\right)_r}{(1+\nu)_r} \frac{(bz)^r}{r!}$$

$$= \sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{2}]} \frac{n! t^k (-k)_{2r}}{k!(1+y)^k (n-k)! 2^{2r} (1+\nu)_r} \frac{(bz)^r}{r!} \tag{2.3.1}$$

Replacing k by $k+2r$ in equation (2.3.1) and applying series manipulation formula (1.22.9), we get

$$\Omega = \sum_{k=0}^{n}\sum_{r=0}^{[\frac{n-k}{2}]} \frac{n! t^{k+2r} (-k-2r)_{2r}}{(k+2r)!(1+y)^{k+2r} (n-k-2r)! 2^{2r} (1+\nu)_r} \frac{(bz)^r}{r!} \tag{2.3.2}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k t^k (-n)_k}{(1+y)^k k!} \sum_{r=0}^{[\frac{n-k}{2}]} \frac{\left(\frac{-n+k}{2}\right)_r \left(\frac{-n+k+1}{2}\right)_r}{(1+\nu)_r (1+y)^{2r}} \frac{(t)^{2r}(bz)^r}{r!} \tag{2.3.3}$$

Now write inner summation of (2.3.3) in Hypergeometric notation, we obtain the generating relation (2.2.1)

2.4 Some Summations Formulae

Case - I: Put $b = 1$, $t = -2$ and $y = z$ in both sides of generating relation (2.2.1), we get

$$\sum_{k=0}^{n} \frac{(-2)^k n!}{(1+z)^k k!(n-k)!} {}_2F_1\left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; z\right]$$

$$= \sum_{k=0}^{n} \frac{(2)^k (-n)_k}{(1+z)^k k!} {}_2F_1\left[\frac{-n+k}{2}, \frac{-n+k+1}{2}; 1+\nu; \frac{4z}{(1+z)^2}\right] \tag{2.4.1}$$

Suppose the left hand side of (2.4.1) is denoted by Ψ, then

$$\Psi = \sum_{k=0}^{n} \frac{(2)^k (-n)_k}{(1+z)^k k!} {}_2F_1\left[\frac{-n+k}{2}, \frac{-n+k+1}{2}; 1+\nu; \frac{4z}{(1+z)^2}\right] \tag{2.4.2}$$

Now applying Goursat's quadratic transformation (1.21.7) in right hand side of (2.4.2), we obtain

$$\Psi = \sum_{k=0}^{n} \frac{(2)^k (-n)_k}{(1+z)^n k!} {}_2F_1[-n+k, -n+k-\nu; 1+\nu; z]$$

$$= \sum_{k=0}^{n}\sum_{r=0}^{n-k} \frac{2^k (-n)_k (-n+k)_r (-n+k-\nu)_r}{(1+z)^n k! (1+\nu)_r} \frac{z^r}{r!}$$

$$= \frac{1}{(1+z)^n} \sum_{r,k=0}^{r+k\leq n} \frac{(-n)_{r+k} (-\nu-n)_{r+k}}{(1+\nu)_r (-\nu-n)_k} \frac{z^r}{r!} \frac{2^k}{k!} = \frac{1}{(1+z)^n} F_4\left[-n, -\nu-n; 1+\nu, -\nu-n; z, 2\right] \tag{2.4.3}$$

Now applying a reduction formula (1.21.21) of Bailey in (2.4.3) for Appell's function of fourth kind F_4, after simplification, we have (2.2.2) and (2.2.3).

Case - II: Put $b = -1$, $t = -2$ and $y = z$ in both sides of (2.2.1), we get

$$\sum_{k=0}^{n} \binom{n}{k} {}_2F_1 \left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; -z \right] \frac{(-2)^k}{(1+z)^k}$$

$$= \sum_{k=0}^{n} \frac{(2)^k(-n)_k}{(1+z)^k k!} {}_2F_1 \left[\frac{-n+k}{2}, \frac{-n+k+1}{2}; 1+\nu; \frac{-4z}{(1+z)^2} \right] \tag{2.4.4}$$

Now equating right hand sides of (2.4.4) and (2.1.2), we have a new summation formula (2.2.4).

Case - III: Put $b = 1$, $t = -2$ and $y = -z$ in both sides of (2.2.1), we have another summation formula

$$\sum_{k=0}^{n} \binom{n}{k} {}_2F_1 \left[\frac{-k}{2}, \frac{-k+1}{2}; 1+\nu; z \right] \frac{(-2)^k}{(1-z)^k}$$

$$= \sum_{k=0}^{n} \frac{(2)^k(-n)_k}{(1-z)^k k!} {}_2F_1 \left[\frac{-n+k}{2}, \frac{-n+k+1}{2}; 1+\nu; \frac{4z}{(1-z)^2} \right] \tag{2.4.5}$$

2.5 Bilinear and Bilateral Generating Relations using Fractional Derivative

Consider the generating relation of Feldheim [113, p.120 (12)] in the form

$$\sum_{n=0}^{\infty} \frac{1}{(1+c)_n} P_n^{(c,a-n)}(x) t^n = exp\left\{ \frac{(1+x)t}{2} \right\} {}_1F_1 \left[\begin{matrix} -a \ ; & \frac{(1-x)t}{2} \\ 1+c; & \end{matrix} \right] \tag{2.5.1}$$

where $P_n^{(c,a-n)}(x)$ and ${}_1F_1$ are restricted Jacobi's polynomials and Kummer's confluent hypergeometric function [252, p.123 (1)], respectively.

In equation (2.5.1), replacing t by bt, multiplying both the sides by $(1-by)^m b^{d-1}$,(m being a positive integer), using the operator $D_b^{(d-e)}$ on both the sides and interpreting the result with the help of the definition (1.2.26), we get the bilateral generating relation in the form

$$\sum_{n=0}^{\infty} \frac{(d)_n P_n^{(c,a-n)}(x)}{(e)_n(1+c)_n} {}_2F_1 \left[\begin{matrix} -m, d+n; & \\ e+n \ \ ; & by \end{matrix} \right] (bt)^n$$

$$= F^{(3)} \left[\begin{array}{l} d :: -; -; - : -; -a \quad ; -m; \\ e :: -; -; - : -; 1+c; \quad -; \end{array} \frac{b(1+x)t}{2}, \frac{b(1-x)t}{2}, by \right] \tag{2.5.2}$$

where ${}_2F_1$ and $F^{(3)}$ are Gauss's ordinary hypergeometric polynomial [252, p.45 (1)] and Srivastava's triple hypergeometric function respectively.

In (2.5.2), replacing y, d, e and b by $\frac{1-y}{2}$, $1+d+e+m$, $1+e$ and 1, respectively and using the definition (1.23.38) of Jacobi's polynomial, we get a bilinear generating relation for Jacobi's polynomials in the following form

$$\sum_{n=0}^{\infty} \frac{m!(1+d+e+m)_n P_n^{(c,a-n)}(x) P_m^{(e+n,d)}(y)}{(1+e)_{m+n}(1+c)_n} t^n$$

$$= F^{(3)} \left[\begin{array}{l} 1+d+e+m :: -; -; - : -; -a \quad ; -m; \\ 1+e \quad :: -; -; - : -; 1+c; \quad -; \end{array} \frac{(1+x)t}{2}, \frac{(1-x)t}{2}, \frac{1-y}{2} \right] \tag{2.5.3}$$

2.6 Linear, Bilinear and Bilateral Generating Relations using the Process of Confluence

In (2.5.2) setting $b = 1$, replacing x and t by $\left(\frac{2x}{c} - 1\right)$ and $-(1+c)t$, respectively, taking $|c| \longrightarrow \infty$, using the confluence principle [184, p.49 (3.5.6); see also 13, pp.333-334 (5.6); 322, p.103 (5.3.4)], we get

$$\sum_{n=0}^{\infty} \frac{(d)_n L_n^{(a-n)}(x)}{(e)_n} {}_2F_1 \left[\begin{array}{l} -m, d+n; \\ e+n; \end{array} y \right] t^n$$

$$= {}_3\Phi_D^{(1)} [d, -a, -m; e; -t, y, -xt] \tag{2.6.1}$$

$$= \Phi_D^{(3)} [d; -a, -m, -; e; -t, y, -xt] \tag{2.6.2}$$

$$= F_{D_1} [d, d, d; -a, -m, -; e, e, e; -t, y, -xt] \tag{2.6.3}$$

Here $L_n^{(a-n)}(x)$ are the restricted Laguerre's polynomials (1.23.39).

In (2.6.1) or (2.6.2) or (2.6.3), replacing e, t and y by $1+e$, $\frac{t}{d}$ and $\frac{y}{d}$, respectively and taking $|d| \longrightarrow \infty$, we get a bilinear generating function for restricted Laguerre's polynomials

$$\sum_{n=0}^{\infty} \frac{m! L_n^{(a-n)}(x) L_m^{(e+n)}(y)}{(1+e)_{m+n}} t^n = \Phi_3^{(3)} [-a, -m; 1+e; -t, y, -xt] \tag{2.6.4}$$

Setting $t=-y$ in (2.6.3) and using a transformation of Exton[98, p.91], we get

$$\sum_{n=0}^{\infty} \frac{(d)_n L_n^{(a-n)}(x)}{(e)_n} {}_2F_1 \left[\begin{array}{c} -m, d+n; \\ e+n \quad ; \end{array} y \right] (-y)^n = \Phi_1 \left[d; -(a+m); e; y, xy\right] \tag{2.6.5}$$

Replacing y by $-y$ and taking $m=0$, (2.6.5) reduces to a known generating function of Khan[156, p.183 (4.6)]

$$\sum_{n=0}^{\infty} \frac{(d)_n L_n^{(a-n)}(x)}{(e)_n} y^n = \Phi_1 \left[d; -a; e; -y, -xy\right] \tag{2.6.6}$$

When y is replaced by $\frac{y}{d}$, taking $|d| \longrightarrow \infty$, (2.6.6) reduces to another known generating function of Khan [153, p.439 (3.1)]

$$\sum_{n=0}^{\infty} \frac{L_n^{(a-n)}(x)}{(e)_n} y^n = \Phi_3 \left[-a; e; -y, -xy\right] \tag{2.6.7}$$

When $y=0$ or $m=0$, (2.6.3) reduces to (2.6.6).

When $x=0$, (2.6.3) reduces to

$$\sum_{n=0}^{\infty} \frac{(a)_n (d)_n}{(e)_n} {}_2F_1 \left[\begin{array}{c} -m, d+n; \\ e+n \quad ; \end{array} y \right] \frac{t^n}{n!} = F_1 \left[d; a; -m; e; t, y\right] \tag{2.6.8}$$

Replacing t by $\frac{t}{a}$ in (2.6.8) and taking $|a| \longrightarrow \infty$, we get

$$\sum_{n=0}^{\infty} \frac{(d)_n}{(e)_n} {}_2F_1 \left[\begin{array}{c} -m, d+n; \\ e+n \quad ; \end{array} y \right] \frac{t^n}{n!} = \Phi_1 \left[d; -m; e; y, t\right] \tag{2.6.9}$$

On replacing y, t and e by $\frac{y}{d}$, $\frac{t}{d}$ and $1+e$, respectively and taking $|d| \longrightarrow \infty$, (2.6.8) reduces to

$$\sum_{n=0}^{\infty} \frac{m!(a)_n}{(1+e)_{m+n}} L_m^{(e+n)}(y) \frac{t^n}{n!} = \Phi_2 \left[a, -m; 1+e; t, y\right] \tag{2.6.10}$$

Similarly by the process of confluence, (2.6.10) gives

$$\sum_{n=0}^{\infty} \frac{m!}{(1+e)_{m+n}} L_m^{(e+n)}(y) \frac{t^n}{n!} = \Phi_3 \left[-m; 1+e; y, t\right] \tag{2.6.11}$$

When $y=0$ or $m=0$ in (2.6.4), we again get (2.6.7) and when $x=0$, (2.6.4) reduces to (2.6.10). Alternatively (2.6.1), (2.6.8) and (2.6.9) can also be written in the following bilateral and linear generating relations

$$\sum_{n=0}^{\infty} \frac{(1+e+d+m)_n}{(1+e+m)_n} L_n^{(a-n)}(x) P_m^{(e+n,d)}(y) t^n$$

$$= \frac{(1+e)_m}{m!} {}_3\Phi_D^{(1)} \left[1+e+d+m, -a, -m; 1+e; -t, \frac{1-y}{2}, -xt \right] \qquad (2.6.12)$$

$$\sum_{n=0}^{\infty} \frac{(a)_n (1+e+d+m)_n}{(1+e+m)_n} P_m^{(e+n,d)}(y) \frac{t^n}{n!} = \frac{(1+e)_m}{m!} F_1 \left[1+e+d+m, a, -m; 1+e; t, \frac{1-y}{2} \right] \qquad (2.6.13)$$

and

$$\sum_{n=0}^{\infty} \frac{(1+e+d+m)_n}{(1+e+m)_n} P_m^{(e+n,d)}(y) \frac{t^n}{n!} = \frac{(1+e)_m}{m!} \Phi_1 \left[1+e+d+m; -m; 1+e; \frac{1-y}{2}, t \right] \qquad (2.6.14)$$

Chapter 3

Some Generating Relations Involving Multi-Variable Laguerre Polynomials

3.1 Introduction

In this chapter deals with m-variable polynomial sets generated by the functions in the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt)$ and two generating relations are obtained as applications of general theorems associated with multiple series identities.

If m-variable polynomial sets $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$ has a generating function of the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt)$ given by equation (3.4.1) then theorem 2 yields another generating function in the form $(1-t)^{-c}G\left(\frac{x_1t}{1-t}, \frac{x_2t}{1-t}, \frac{x_3t}{1-t}, \ldots, \frac{x_mt}{1-t}\right)$ given by equation (3.4.3) for same polynomial set $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$.

3.2 Partial Differential Equations Associated with Polynomial Sets

Theorem 1

If

$$e^t\phi_1(x_1t)\phi_2(x_2t)\ldots\phi_m(x_mt) = \sum_{n=0}^{\infty}\sigma_n(x_1, x_2, \ldots, x_m)t^n \tag{3.2.1}$$

then

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots + x_m\frac{\partial}{\partial x_m}\right)\sigma_0(x_1, x_2, \ldots, x_m) = 0 \tag{3.2.2}$$

and for $n \geq 1$

$$(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \cdots + x_m\frac{\partial}{\partial x_m})\sigma_n(x_1, x_2, \ldots, x_m) - n\ \sigma_n(x_1, x_2, \ldots, x_m)$$

$$= -\sigma_{n-1}(x_1, x_2, \ldots, x_m) \tag{3.2.3}$$

Proof:- Let us consider the generating relation of the type

$$e^t\phi_1(x_1t)\phi_2(x_2t)\ldots\phi_m(x_mt) = \sum_{n=0}^{\infty}\sigma_n(x_1, x_2, \ldots, x_m)t^n \tag{3.2.4}$$

Suppose

$$F = e^t\phi_1(x_1t)\phi_2(x_2t)\ldots\phi_m(x_mt) \tag{3.2.5}$$

or

$$F = e^t \phi_1 \phi_2 \ldots \phi_m$$

Then

$$\frac{\partial F}{\partial x_1} = t\ e^t \phi_1' \phi_2 \phi_3 \ldots \phi_m \tag{3.2.6}$$

$$\frac{\partial F}{\partial x_2} = te^t \phi_1 \phi_2' \phi_3 \ldots \phi_m \tag{3.2.7}$$

$$\frac{\partial F}{\partial x_3} = te^t \phi_1 \phi_2 \phi_3' \ldots \phi_m \tag{3.2.8}$$

similarly

$$\frac{\partial F}{\partial x_m} = te^t \phi_1 \phi_2 \phi_3 \ldots \phi_m' \tag{3.2.9}$$

and

$$\frac{\partial F}{\partial t} = e^t \phi_1 \phi_2 \phi_3 \ldots \phi_m + x_1 e^t \phi_1' \phi_2 \phi_3 \ldots \phi_m + x_2 e^t \phi_1 \phi_2' \phi_3 \ldots \phi_m$$
$$+ x_3 e^t \phi_1 \phi_2 \phi_3' \ldots \phi_m + \ldots + x_m e^t \phi_1 \phi_2 \phi_3 \ldots \phi_m' \tag{3.2.10}$$

Eliminating $\phi_1, \phi_1', \phi_2, \phi_2', \phi_3, \phi_3', \ldots\ldots, \phi_m$ and ϕ_m' from the above equations, we obtain

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + \ldots + x_m \frac{\partial}{\partial x_m} \right) F - t \frac{\partial F}{\partial t} = -tF \tag{3.2.11}$$

From equations (3.2.4) and (3.2.11), we have

$$\sum_{n=0}^{\infty} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + \ldots + x_m \frac{\partial}{\partial x_m} \right) \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n$$
$$- \sum_{n=1}^{\infty} n\ \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n = - \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^{n+1}$$
$$= - \sum_{n=1}^{\infty} \sigma_{n-1}(x_1, x_2, x_3, \ldots, x_m) t^n$$

Now equating the coefficients of like powers of t, we get (3.2.2) and (3.2.3).

3.3 Series Representation of Polynomials

The series representation of $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$ is given by

$$\sigma_n(x_1, x_2, x_3, \ldots, x_m)$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_m=0}^{n-k_1-k_2-k_3-\ldots-k_{m-1}} \frac{a_1(k_1)\ a_2(k_2)\ a_3(k_3)\ \ldots a_m(k_m) {x_1}^{k_1} {x_2}^{k_2} {x_3}^{k_3} \ldots {x_m}^{k_m}}{(n-k_1-k_2-k_3-\ldots-k_m)!}$$

$$= \sum_{k_1,k_2,k_3,\ldots,k_m=0}^{k_1+k_2+k_3+\ldots+k_m \le n} \frac{(-n)_{k_1+k_2+k_3+\ldots+k_m} a_1(k_1)\ a_2(k_2)\ a_3(k_3)\ \ldots a_m(k_m)}{n!} \times$$

$$\times (-x_1)^{k_1} (-x_2)^{k_2} (-x_3)^{k_3} \ldots (-x_m)^{k_m} \tag{3.3.1}$$

where $\{a_1(k_1)\}$, $\{a_2(k_2)\}$, $\{a_3(k_3)\}$, $\ldots, \{a_m(k_m)\}$ are bounded sequences of arbitrary real or complex numbers, $\forall \quad k_j \ \epsilon \ \{0, 1, 2, \ldots\}; \qquad 1 \le j \le m.$

Proof:- Suppose the functions $\phi_1, \phi_2,\ \phi_3$,..........,ϕ_m in (3.2.1) have the formal power-series expansions.

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty} a_1(k_1) u_1^{k_1}; \qquad a_1(0) \neq 0 \tag{3.3.2}$$

$$\phi_2(u_2) = \sum_{k_2=0}^{\infty} a_2(k_2) u_2^{k_2}; \qquad a_2(0) \neq 0 \tag{3.3.3}$$

$$\phi_3(u_3) = \sum_{k_3=0}^{\infty} a_3(k_3) u_3^{k_3}; \qquad a_3(0) \neq 0 \tag{3.3.4}$$

and

$$\phi_m(u_m) = \sum_{k_m=0}^{\infty} a_m(k_m) u_m^{k_m}; \qquad a_m(0) \neq 0 \tag{3.3.5}$$

Then (3.2.1) yields

$$\sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3, \ldots, x_m) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k_1,k_2,k_3,\ldots,k_m=0}^{\infty} a_1(k_1) x_1^{k_1} a_2(k_2) x_2^{k_2} \ldots a_m(k_m) x_m^{k_m} \frac{t^{n+k_1+k_2+k_3+\ldots+k_m}}{n!} \tag{3.3.6}$$

Now using multiple series identity (1.22.7), we get

$$\sum_{n=0}^{\infty} \{\sigma_n(x_1, x_2, x_3, \ldots, x_m)\} t^n$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_m=0}^{n-k_1-k_2-\ldots-k_{m-1}} \frac{a_1(k_1) a_2(k_2) \ldots a_m(k_m) {x_1}^{k_1} {x_2}^{k_2} \ldots {x_m}^{k_m}}{(n-k_1-k_2\ldots-k_m)!} \right\} t^n \tag{3.3.7}$$

Now comparing the coefficients of t^n in equation (3.3.7), we get the series representation (3.3.1) for $\sigma_n(x_1, x_2, x_3, \ldots, x_m)$.

3.4 General Theorem Associated with Generating Relation and Generating Function

Theorem 2

If

$$e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)\ldots\phi_m(x_mt) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3,\ldots,x_m)t^n \tag{3.4.1}$$

and

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty} a_1(k_1)u_1^{k_1}; \qquad \phi_2(u_2) = \sum_{k_2=0}^{\infty} a_2(k_2)u_2^{k_2};$$

$$\phi_3(u_3) = \sum_{k_3=0}^{\infty} a_3(k_3)u_3^{k_3}; \qquad \ldots\phi_m(u_m) = \sum_{k_m=0}^{\infty} a_m(k_m)u_m^{k_m}; \tag{3.4.2}$$

where $a_1(0) \neq 0, \quad a_2(0) \neq 0, \quad a_3(0) \neq 0, \ldots a_m(0) \neq 0$

then

$$(1-t)^{-c}G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\frac{x_3t}{1-t},\ldots,\frac{x_mt}{1-t}\right) = \sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,x_3,\ldots,x_m)t^n \tag{3.4.3}$$

where c is arbitrary and

$$G(w_1,w_2,w_3,\ldots,w_m)$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}(c)_{k_1+k_2+k_3+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ a_3(k_3)\ \ldots a_m(k_m)w_1^{k_1}w_2^{k_2}w_3^{k_3}\ldots w_m^{k_m} \tag{3.4.4}$$

where $\{a_1(k_1)\}, \ \{a_2(k_2)\}, \ \{a_3(k_3)\}, \ \ldots, \{a_m(k_m)\}$ are bounded sequences of arbitrary real or complex numbers, $\forall \quad k_j \ \epsilon \ \{0,1,2,\ldots\}; \qquad 1 \leq j \leq m.$

Proof:-Now consider the the new series in the form

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,x_3,\ldots,x_m)t^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_m=0}^{n-k_1-k_2-k_3-\ldots-k_{m-1}}\frac{(c)_na_1(k_1)a_2(k_2)\ldots a_m(k_m){x_1}^{k_1}{x_2}^{k_2}\ldots {x_m}^{k_m}}{(n-k_1-k_2-\ldots-k_m)!}\right)t^n \tag{3.4.5}$$

Now using multiple series identity (1.22.8), we get

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,\ldots,x_m)t^n$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}\frac{(c)_{n+k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}}{(n)!}\right)t^n \tag{3.4.6}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(c+k_1+k_2+\ldots+k_m)_n}{n!}t^n \tag{3.4.7}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}\times$$

$$\times\ {}_1F_0\left[\begin{array}{ll} c+k_1+k_2+\ldots+k_m\ ; & \\ - \ ; & t\end{array}\right] \tag{3.4.8}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ \ldots a_m(k_m)\ (x_1t)^{k_1}(x_2t)^{k_2}\ldots(x_mt)^{k_m}\times$$

$$\times\ (1-t)^{-(c+k_1+k_2+\ldots+k_m)} \tag{3.4.9}$$

Therefore

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,\ldots,x_m)t^n=(1-t)^{-c}\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\,a_2(k_2)\ \ldots a_m(k_m)\times$$

$$\times\left(\frac{x_1t}{1-t}\right)^{k_1}\left(\frac{x_2t}{1-t}\right)^{k_2}\ldots\left(\frac{x_mt}{1-t}\right)^{k_m} \tag{3.4.10}$$

$$=(1-t)^{-c}G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\ldots,\frac{x_mt}{1-t}\right)$$

where $G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\ldots,\frac{x_mt}{1-t}\right)$ is defined by the equation (3.4.4).

3.5 Applications of Theorems 1 and 2 in Generating Relations

If $\phi_1(u_1),\phi_2(u_2),\phi_3(u_3)\ldots\phi_m(u_m)$, are specified in hypergeometric forms, then theorem 2 gives for polynomial sets $\sigma_n(x_1,x_2,x_3,\ldots,x_m)$, a class of generating relation involving m-variable hypergeometric polynomials.

Now apply theorems 1 and 2 to Laguerre polynomials of m-variable $L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)$ defined explicitly by (1.23.36)

$$L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n\ldots(1+\alpha_m)_n}{(n!)^m}\times$$

$$\times\, \Psi_2^{(m)}\left[-n;1+\alpha_1,1+\alpha_2,\ldots,1+\alpha_m;x_1,x_2,\ldots,x_m\right] \tag{3.5.1}$$

Now consider a polynomial set in the form

$$\sigma_n(x_1,x_2,\ldots,x_m) = \frac{(n!)^{m-1}L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)}{(1+\alpha_1)_n(1+\alpha_2)_n\ldots(1+\alpha_m)_n} \tag{3.5.2}$$

and

$$\phi_1(u_1) = {}_0F_1(-;1+\alpha_1;-u_1) = \sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}{u_1}^{k_1}}{k_1!(1+\alpha_1)_{k_1}} \tag{3.5.3}$$

$$\phi_2(u_2) = {}_0F_1(-;1+\alpha_2;-u_2) = \sum_{k_2=0}^{\infty}\frac{(-1)^{k_2}{u_2}^{k_2}}{k_2!(1+\alpha_2)_{k_2}} \tag{3.5.4}$$

$$\phi_3(u_3) = {}_0F_1(-;1+\alpha_3;-u_3) = \sum_{k_3=0}^{\infty}\frac{(-1)^{k_3}{u_3}^{k_3}}{k_3!(1+\alpha_3)_{k_3}} \tag{3.5.5}$$

similarly

$$\phi_m(u_m) = {}_0F_1(-;1+\alpha_m;-u_m) = \sum_{k_m=0}^{\infty}\frac{(-1)^{k_m}{u_m}^{k_m}}{k_m!(1+\alpha_m)_{k_m}} \tag{3.5.6}$$

Then

$$a_1(k_1) = \frac{(-1)^{k_1}}{k_1!(1+\alpha_1)_{k_1}}, a_2(k_2) = \frac{(-1)^{k_2}}{k_2!(1+\alpha_2)_{k_2}},\ldots,a_m(k_m) = \frac{(-1)^{k_m}}{k_m!(1+\alpha_m)_{k_m}} \tag{3.5.7}$$

Then generating relation is given by

$$e^t{}_0F_1(-;1+\alpha_1;-x_1t){}_0F_1(-;1+\alpha_2;-x_2t)\ldots{}_0F_1(-;1+\alpha_m;-x_mt)$$

$$= \sum_{n=0}^{\infty}\frac{(n!)^{m-1}L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)}{(1+\alpha_1)_n(1+\alpha_2)_n\ldots(1+\alpha_m)_n}t^n \tag{3.5.8}$$

We use theorem-1 to conclude that

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\ldots+x_m\frac{\partial}{\partial x_m}\right)L_0^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m) = 0$$

and for $n \geq 1$,

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\ldots+x_m\frac{\partial}{\partial x_m}\right)L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)$$

$$-n\,L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)=-\frac{(\alpha_1+n)(\alpha_2+n)\ldots(\alpha_m+n)}{n^{m-1}}L_{n-1}^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)$$

In applying theorem-2 to Laguerre polynomials of m variables $L_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)$, and

$$G(w_1,w_2,\ldots,w_m)$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}(c)_{k_1+k_2+\ldots+k_m}a_1(k_1)\ a_2(k_2)\ldots a_m(k_m){w_1}^{k_1}{w_2}^{k_2}\ldots{w_m}^{k_m}$$

$$=\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}\frac{(c)_{k_1+k_2+\cdots+k_m}(-1)^{k_1+k_2+\cdots+k_m}{w_1}^{k_1}{w_2}^{k_2}\ldots{w_m}^{k_m}}{k_1!k_2!\ldots k_m!(1+\alpha_1)_{k_1}(1+\alpha_2)_{k_2}\ldots(1+\alpha_m)_{k_m}}$$

$$=\psi_2^{(m)}\left[c;1+\alpha_1,1+\alpha_2,\ldots,1+\alpha_m;-w_1,-w_2,\ldots,-w_m\right] \qquad (3.5.9)$$

Therefore theorem-2 yields

$$(1-t)^{-c}\psi_2^{(m)}\left[c;1+\alpha_1,1+\alpha_2,\ldots,1+\alpha_m;\frac{-x_1t}{1-t},\frac{-x_2t}{1-t},\ldots,\frac{-x_mt}{1-t}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(n!)^{m-1}(c)_nL_n^{(\alpha_1,\alpha_2,\ldots,\alpha_m)}(x_1,x_2,\ldots,x_m)}{(1+\alpha_1)_n(1+\alpha_2)_n\ldots(1+\alpha_m)_n}t^n \qquad (3.5.10)$$

Chapter 4

A General Family of Generating Functions

4.1 Introduction

In this chapter we derive two general theorems on generating relations for a certain sequence of functions. The generating functions for multivariable generalized hypergeometric polynomials are shown here as special cases of a general class of generating relations. A number of results associated with Jacobi, Khandekar polynomials of several variables and other results of multiple Gaussian hypergeometric functions scattered in the literature of special functions follow as applications of main results.

It has been observed that the generating functions play a remarkable role in the study of the polynomial sets. Therefore, an attempt has been made to establish some interesting generating relations for generalized Hypergeometric polynomials by making use of series iteration techniques in the present study.

4.2 General Multiple Series Identities

Theorem 1: Let $\{S_q(n, k_1, k_2, \ldots, k_m)\}$, $q = 1, 2, 3$ be bounded multiple sequences of arbitrary real or complex numbers, for every n, $k_r \in \{0, 1, 2, \ldots\}$; $r = 1, 2, \ldots, m$. Also let $(\lambda)_n$ type notations denote the Pochhammer symbol defined by (1.2.11) and $z_1, z_2, \ldots, z_m$ are several complex variables; $I_1, I_2, \ldots, I_m$ are arbitrary positive integers, then

$$\sum_{n=0}^{\infty} S_1(n) \sum_{k_1,k_2,\ldots,k_m=0}^{L\leq n} (-n)_L\, S_2(k_1, k_2, \ldots, k_m) \prod_{j=1}^{m} \left\{ (\theta_j + n)_{k_j} \frac{{z_j}^{k_j}}{(k_j)!} \right\} \frac{t^n}{n!}$$

$$= \sum_{n,k_1,k_2,\ldots,k_m=0}^{\infty} S_1(n+L)\, S_2(k_1, k_2, \ldots, k_m) \prod_{j=1}^{m} \left\{ \frac{(\theta_j)_{n+L+k_j} (-t)^{I_j k_j}\, {z_j}^{k_j}}{(\theta_j)_{n+L}\, (k_j)!} \right\} \frac{t^n}{n!} \tag{4.2.1}$$

provided that each of the multiple series involved in (4.2.1) converges absolutely and $L = I_1 k_1 + I_2 k_2 + \cdots + I_m k_m$.

Theorem 2: Under the hypotheses stated of theorem 1, we have

$$\sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_m=0}^{L\le n}\frac{S_3(n-L)(\lambda+n-L)_{k_1+k_2+\ldots,+k_m}}{(n-L)!}\Big\{\prod_{j=1}^{m}\frac{{z_j}^{k_j}}{(k_j)!}\Big\}t^n$$

$$=\Big(1-\sum_{j=1}^{m}z_j\ t^{I_j}\Big)^{-\lambda}\sum_{n=0}^{\infty}\frac{S_3(n)}{n!}\left(\frac{t}{1-\sum\limits_{j=1}^{m}z_j\ t^{I_j}}\right)^n \tag{4.2.2}$$

provided that each of the multiple series involved in (4.2.2) converges absolutely and $L = I_1k_1 + I_2k_2 + \cdots + I_mk_m$.

It is important to note that theorems proved here, are of very general nature.

Proof of (4.2.1): The L.H.S. of equation (4.2.1) can be written as

$$T=\sum_{n=0}^{\infty}S_1(n)\sum_{k_1,k_2,\ldots,k_m=0}^{I_1k_1+I_2k_2+\cdots+I_mk_m\le n}(-n)_{I_1k_1+I_2k_2+\cdots+I_mk_m}S_2(k_1,k_2,\ldots,k_m)\times$$

$$\times\,(\theta_1+n)_{k_1}\frac{z_1^{k_1}}{(k_1)!}(\theta_2+n)_{k_2}\frac{z_2^{k_2}}{(k_2)!}\cdots(\theta_m+n)_{k_m}\frac{z_m^{k_m}}{(k_m)!}\frac{t^n}{n!} \tag{4.2.3}$$

Replacing n by $n+I_1k_1+I_2k_2+\cdots+I_mk_m$ in (4.2.3) and using the lemma [326; p.102 (17)]

$$\sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_m=0}^{I_1k_1+I_2k_2+\cdots+I_mk_m\le n}A(n;k_1,k_2,\ldots,k_m)$$

$$=\sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}A(n+I_1k_1+I_2k_2+\cdots+I_mk_m;k_1,k_2,\ldots,k_m) \tag{4.2.4}$$

we get

$$T=\sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_m=0}^{\infty}S_1(n+I_1k_1+I_2k_2+\cdots+I_mk_m)S_2(k_1,k_2,\ldots,k_m)\times$$

$$\times(\theta_1+n+I_1k_1+I_2k_2+\cdots+I_mk_m)_{k_1}\frac{(-t)^{I_1k_1}{z_1}^{k_1}}{(k_1)!}(\theta_2+n+I_1k_1+I_2k_2+\cdots+I_mk_m)_{k_2}\frac{(-t)^{I_2k_2}{z_2}^{k_2}}{(k_2)!}\times$$

$$\times\cdots(\theta_m+n+I_1k_1+I_2k_2+\cdots+I_mk_m)_{k_m}\frac{(-t)^{I_mk_m}{z_m}^{k_m}}{(k_m)!}\frac{t^n}{n!} \tag{4.2.5}$$

Now using the definition of Pochhammer symbol $\Gamma(\alpha+n)=\Gamma(\alpha)(\alpha)_n$, we get the right hand side of (4.2.1).

This identity completes the proof of theorem (4.2.1) under the assumption that the series involved, are absolutely convergent.

Proof of (4.2.2): The L.H.S. of equation (4.2.2) can be written as

$$R = \sum_{n=0}^{\infty} \sum_{k_1,k_2,\ldots,k_m=0}^{I_1k_1+I_2k_2+\cdots+I_mk_m \le n} S_3(n - I_1k_1 - I_2k_2 - \cdots - I_mk_m)\times$$

$$\times \frac{(\lambda + n - I_1k_1 - I_2k_2 - \cdots - I_mk_m)_{k_1+k_2+\cdots+k_m}\, {z_1}^{k_1}\, {z_2}^{k_2}\, \cdots\, {z_m}^{k_m}}{(n - I_1k_1 - I_2k_2 - \cdots - I_mk_m)!\,(k_1)!(k_2)!\cdots(k_m)!} t^n \qquad (4.2.6)$$

Replacing n by $n + I_1k_1 + I_2k_2 + \cdots + I_mk_m$ in (4.2.6) and using the lemma (4.2.4), we get

$$R = \sum_{n=0}^{\infty} S_3(n)\frac{t^n}{n!} \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} \frac{(\lambda+n)_{k_1+k_2+\cdots+k_m}(z_1t^{I_1})^{k_1}(z_2t^{I_2})^{k_2}\cdots(z_mt^{I_m})^{k_m}}{(k_1)!(k_2)!\cdots(k_m)!} \qquad (4.2.7)$$

Now using the reduction formula [326, p.52 (3)], we get

$$R = \sum_{n=0}^{\infty} S_3(n)\frac{t^n}{n!} \sum_{p=0}^{\infty} \frac{(\lambda+n)_p[z_1t^{I_1} + z_2t^{I_2} + \cdots + z_mt^{I_m}]^p}{p!}$$

$$= \sum_{n=0}^{\infty} S_3(n)\frac{t^n}{n!}(1 - z_1t^{I_1} - z_2t^{I_2} - \cdots - z_mt^{I_m})^{-\lambda-n}$$

$$= (1 - z_1t^{I_1} - z_2t^{I_2} - \cdots - z_mt^{I_m})^{-\lambda} \sum_{n=0}^{\infty} \frac{S_3(n)}{n!}\left(\frac{t}{1 - z_1t^{I_1} - z_2t^{I_2} - \cdots - z_mt^{I_m}}\right)^n$$

which is the right hand side of (4.2.2).

4.3 Applications

It is very interesting that formulae proved here, is leading to certain generalizations of the various well-known results due to Shrivastava, Rice et-al.

(i) Setting $m = 3, I_1 = I_2 = I_3 = 1, z_1 = x, z_2 = y, z_3 = z, \theta_1 = 1 + \alpha_1 + \beta_1$, $\theta_2 = 1 + \alpha_2 + \beta_2, \theta_3 = 1 + \alpha_3 + \beta_3$

$$S_1(n) = (1 + \alpha_1 + \beta_1)_n(1 + \alpha_2 + \beta_2)_n(1 + \alpha_3 + \beta_3)_n$$

and

$$S_2(k_1, k_2, k_3) = \frac{(\nu_1)_{k_1}(\nu_2)_{k_2}(\nu_3)_{k_3}}{(\sigma_1)_{k_1}(1+\alpha_1)_{k_1}(\sigma_2)_{k_2}(1+\alpha_2)_{k_2}(\sigma_3)_{k_3}(1+\alpha_3)_{k_3}}$$

in theorem 1, interpreting L.H.S. in terms of generalized Rice polynomials (for r = 3) defined by (1.23.70) and R.H.S. in terms of Srivastava-Daoust function(1.18.1), we get

$$\sum_{n=0}^{\infty}\prod_{i=1}^{3}\left\{\frac{(1+\alpha_i+\beta_i)_n}{(1+\alpha_i)_n}\right\}(n!)^2 H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2;\alpha_3,\beta_3)}\left[\nu_1;\sigma_1:\nu_2;\sigma_2:\nu_3;\sigma_3:x,y,z\right]t^n$$

$$= F^{3:0;1;1;1}_{0:0;2;2;2}\left(\begin{array}{l}[1+\alpha_1+\beta_1:1,2,1,1],\ [1+\alpha_2+\beta_2:1,1,2,1],\ [1+\alpha_3+\beta_3:1,1,1,2]: \\ \overline{\hspace{14cm}}: \end{array}\right.$$

$$\left.\begin{array}{lllll} -; & [\nu_1:1] & ; & [\nu_2:1] & ; \quad [\nu_3:1] \quad ; \\ -; & [1+\alpha_1:1],[\sigma_1:1]; & & [1+\alpha_2:1],\ [\sigma_2:1]; & [1+\alpha_3:1],\ [\sigma_3:1]; \end{array} \; t,\ -xt,\ -yt,\ -zt\right) \tag{4.3.1}$$

(ii) Setting $m=2, I_1=I_2=1, z_1=x, z_2=y, \theta_1=1+\alpha_1+\beta_1, \theta_2=1+\alpha_2+\beta_2,$

$$S_1(n)=(1+\alpha_1+\beta_1)_n$$

and

$$S_2(k_1,k_2)=\frac{(\nu_1)_{k_1}(\nu_2)_{k_2}}{(\sigma_1)_{k_1}(1+\alpha_1)_{k_1}(\sigma_2)_{k_2}(1+\alpha_2)_{k_2}}$$

in theorem 1, interpreting L.H.S. in terms of generalized Rice polynomials (for r = 2) defined by (1.23.70) and using Euler's first linear transformation[252, p.60 (4)] in the R.H.S., we get

$$\sum_{n=0}^{\infty}\left\{\frac{(1+\alpha_1+\beta_1)_n(n!)}{(1+\alpha_1)_n(1+\alpha_2)_n}\right\}H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\left[\nu_1;\sigma_1:\nu_2;\sigma_2:x,y\right]t^n$$

$$=(1-t)^{-1-\alpha_1-\beta_1}F^{3:0;1;0}_{3:0;2;0}\left(\begin{array}{l}[1+\alpha_1+\beta_1:2,2,1],\ [1+\alpha_2+\beta_2:2,1,2],\ [\nu_2:1,0,1]: \\ \\ [1+\alpha_2+\beta_2:2,1,1],\ [1+\alpha_2:1,0,1] \qquad ,\ [\sigma_2:1,0,1]: \end{array}\right.$$

$$\left.\begin{array}{lll} \text{——}; & [\nu_1:1] & ;\text{——}; \\ \text{——}; & [\sigma_1:1],[1+\alpha_1:1]; & \text{——}; \end{array} \; \frac{-yt^2}{(1-t)^2},\frac{-xt}{(1-t)^2},\frac{-yt}{(1-t)}\right) \tag{4.3.2}$$

$$= (1-t)^{-1-\alpha_2-\beta_2} F^{3:0;0;1}_{3:0;0;2}\left(\begin{array}{l} [1+\alpha_2+\beta_2:2,1,2],\ [1+\alpha_1+\beta_1:2,2,1],\ [\nu_1:1,1,0]: \\ \\ [1+\alpha_2+\beta_2:2,1,1],\ [1+\alpha_1:1,1,0] \qquad ,\ \ [\sigma_1:1,1,0]: \end{array}\right.$$

$$\left.\begin{array}{l} \text{——};\text{——};\qquad [\nu_2:1] \qquad ; \\ \qquad\qquad\qquad\qquad\qquad\qquad \frac{-xt^2}{(1-t)^2}, \frac{-xt}{(1-t)}, \frac{-yt}{(1-t)^2} \\ \text{——};\text{——};[\sigma_2:1],[1+\alpha_2:1]; \end{array}\right) \tag{4.3.3}$$

(iii) In equation (4.3.1), replacing x by $\frac{1-x}{2}$, y by $\frac{1-y}{2}$, z by $\frac{1-z}{2}$, $\nu_1=\sigma_1, \nu_2=\sigma_2, \nu_3=\sigma_3$ and using the definition of generalized Jacobi polynomials (r = 3) defined by (1.23.79), we get a known generating relation of Shrivastava [271, p.66 (19)]

$$\sum_{n=0}^{\infty}\prod_{i=1}^{3}\left\{\frac{(1+\alpha_i+\beta_i)_n}{(1+\alpha_i)_n}\right\}(n!)^2 P_n^{(\alpha_1,\beta_1\,;\,\alpha_2,\beta_2\,;\,\alpha_3,\beta_3)}(x,y,z)\,t^n$$

$$= F^{3:0;0;0;0}_{0:0;1;1;1}\left(\begin{array}{l} [1+\alpha_1+\beta_1:1,2,1,1],\ [1+\alpha_2+\beta_2:1,1,2,1],\ [1+\alpha_3+\beta_3:1,1,1,2]: \\ \\ \text{————————————————————————}: \end{array}\right.$$

$$\left.\begin{array}{l} \text{——};\text{————};\text{————};\text{————}; \\ \qquad\qquad\qquad\qquad\qquad\qquad t,\ \frac{(x-1)t}{2},\ \frac{(y-1)t}{2},\ \frac{(z-1)t}{2} \\ \text{——};[1+\alpha_1:1];[1+\alpha_2:1];[1+\alpha_3:1]; \end{array}\right) \tag{4.3.4}$$

(iv) In (theorem 1) setting $m=1$, $I_1=1$, $\theta_1=(1+\alpha+\beta)$, $S_1(n)=(1+\alpha+\beta)_n$, $S_2(k_1)=\frac{(\xi)_{k_1}}{(1+\alpha)_{k_1}(p)_{k_1}}$, $z_1=\nu$, using the definition (1.23.66) of Khandekar polynomials of one variable, we get

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}H_n^{(\alpha,\beta)}[\xi,p,\nu]\,t^n = (1-t)^{-1-\alpha-\beta}\,{}_3F_2\left[\begin{array}{ll} \Delta(2;1+\alpha+\beta),\ \xi & ; \\ & \frac{-4\nu t}{(1-t)^2} \\ 1+\alpha,\quad p & ; \end{array}\right] \tag{4.3.5}$$

which is well known generating function [171, p.159 (4.2)] for the generalized Rice polynomial defined by (1.23.66), where $\Delta(N;a)=\frac{a}{N},\frac{a+1}{N},\cdots,\frac{a+N-1}{N}$. The equation (4.3.5) was obtained by Khandekar [171], using Beta integral technique.

(v) In (theorem 1) setting $m = 1$, $I_1 = 1$, $\theta_1 = (1+\alpha+\beta)$, $S_1(n) = (1+\alpha+\beta)_n$, $S_2(k_1) = \frac{1}{(1+\alpha)_{k_1}}$, $z = \frac{1-x}{2}$, using the definition (1.23.38) of classical Jacobi polynomials, we get:-

$$\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(x)\, t^n = (1-t)^{-1-\alpha-\beta}\, {}_2F_1 \left[\begin{array}{ll} \frac{(1+\alpha+\beta)}{2}, \frac{(2+\alpha+\beta)}{2} & ; \\ & \frac{2(x-1)t}{(1-t)^2} \\ 1+\alpha & ; \end{array} \right] \tag{4.3.6}$$

which is well known generating relation [252; p.256 (10)] for the Jacobi polynomials defined by (1.23.38) and can also be obtained directly from (4.3.4).

(vi) In (theorem 2) setting $m = 2$, $I_1 = I_2 = 1$, putting $S_3(n) = \dfrac{\prod_{i=1}^{A}(a_i)_n}{\prod_{j=1}^{B}(b_j)_n}$, using the double series identity of Srivastava [326, p.52 (2)] and replacing $(z_1 + z_2)$ by z, we get

$$\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_n}{\prod_{j=1}^{B}(b_j)_n}\, {}_{B+2}F_A \left[\begin{array}{ll} -n, 1-\lambda-n, 1-(b_B)-n & ; \\ & (-1)^{A+B} z \\ 1-(a_A)-n & ; \end{array} \right] \frac{t^n}{n!}$$

$$= (1-zt)^{-\lambda}\, {}_AF_B \left[\begin{array}{ll} (a_A) & ; \\ & \frac{t}{(1-zt)} \\ (b_B) & ; \end{array} \right] \tag{4.3.7}$$

which is a particular case of known generating relation of Srivastava[309, p.76 (3.1); see also 326, p.145 (30)].

For different values of $m, I_1, I_2, I_3, \ldots$ and bounded sequences we can derive a number of known and unknown generating relations involving generalized hypergeometric polynomials and Kampé de Fériet function of two variables. Srivastava function $F^{(3)}$ and its special cases, Exton's double hypergeometric functions and multivariable Srivastava-Daoust function.

Chapter 5

Some Generating Relations and Hypergeometric Transformations using Decomposition Technique

5.1 Introduction

In this chapter we obtain four linear generating relations for generalized Hermite polynomials of Dickinson-Warsi and two hypergeometric transformations for Gauss's ordinary hypergeometric function, using Barr's identity and Carlitz's generating relations using following identity.

$$\sum_{m=0}^{\infty} \Phi(m) = \sum_{m=0}^{\infty} \Phi(2m) + \sum_{m=0}^{\infty} \Phi(2m+1) \tag{5.1.1}$$

In 1969, Barr [19, p.591 (1)] gave the following hypergeometric series identity

$$\begin{aligned} {}_AF_B\left[\begin{array}{l}(a_A);\\ (b_B);\end{array} z\right] &= {}_{2A}F_{2B+1}\left[\begin{array}{l}\frac{(a_A)}{2}, \quad \frac{(a_A)+1}{2};\\ \frac{1}{2}, \frac{(b_B)}{2}, \frac{(b_B)+1}{2};\end{array} 4^{(A-B-1)}z^2\right] + \\ &+ z\,\frac{\prod_{m=1}^{A}(a_m)}{\prod_{n=1}^{B}(b_n)}\ {}_{2A}F_{2B+1}\left[\begin{array}{l}\frac{(a_A)+1}{2}, \quad \frac{(a_A)+2}{2};\\ \frac{3}{2}, \frac{(b_B)+1}{2}, \frac{(b_B)+2}{2};\end{array} 4^{(A-B-1)}z^2\right] \end{aligned} \tag{5.1.2}$$

where, for convergence, $A \leq B$ and $|z| < \infty$, or $A = B+1$ and $|z| < 1$, it being assumed that $b_n \neq 0, -1, -2, -3, -4, \ldots \quad \forall \quad n \in \{1, 2, 3, \ldots, B\}$ and (a_A) represents the sequence of parameters given by $a_1, a_2, a_3, \ldots, a_A$.

It is well known that the Hermite polynomials $H_n(x)$ can be entirely reduced to the associated Laguerre polynomials $L_n^{(a)}(x)$ with the parameters $a = \pm\frac{1}{2}$ by means of the formulas [252, p.216 Q.1; 333, p.106 (5.6.1)]

$$H_{2n}(x) = (-1)^n 2^{2n}\ (n)!\ L_n^{(-\frac{1}{2})}(x^2) \tag{5.1.3}$$

$$H_{2n+1}(x) = (-1)^n 2^{2n+1}\ (n)!\ x\ L_n^{(\frac{1}{2})}(x^2) \tag{5.1.4}$$

This relationship of Hermite to Laguerre polynomials is generalized by Dickinson and Warsi [82, p.258 (10), p.259 (12); see also 226, p.352 (4.1), (4.2)] in the form

$$H_{2n}^{(a)}(x) = (-1)^n 2^{2n}\ (n)!\ L_n^{(a)}(x^2) \tag{5.1.5}$$

and

$$H_{2n+1}^{(a)}(x) = (-1)^n 2^{2n+1}\ (n)!\ x\ L_n^{(a+1)}(x^2) \tag{5.1.6}$$

In 1960, Carlitz [31, pp.221-222 (8,10); see also 326, p.420 (20),(21); 118, p.58] gave the following two generating relations

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{m!\ \ n!} L_{n+m}^{(a)}(x) t^n = (1-t)^{-1-a-m} exp\left[\frac{-xt}{1-t}\right] L_m^{(a)}\left[\frac{x}{1-t}\right] \tag{5.1.7}$$

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{m!\ \ n!} L_{m+n}^{(a-n)}(x) t^n = (1+t)^a exp[-xt] L_m^{(a)}[x(1+t)] \tag{5.1.8}$$

It is to be noted that linear generating relations (5.2.1), (5.2.2), (5.2.3), (5.2.4) for even and odd generalized Hermite polynomials of Dickinson-Warsi and two hypergeometric transformations (5.2.5), (5.2.6) do not seem to have been recorded earlier.

In this chapter, we obtain some generating relations for Subuhi Khan polynomial $H_{n,\alpha,\beta}$ and Lagrange's polynomial of two variable $g_n^{(\alpha,\beta)}(x,y)$ by using Decomposition technique by different method.

5.1.1 Hypergeometric Form of Subuhi Khan Polynomial

Subuhi Khan Polynomial [170, p.84 (4.2.8, 4.2.9); see also 220, p.60 (2.9)] is generated by the following generating relation

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x) \frac{t^n}{n!} = exp\left(\frac{xt}{\alpha}\right) exp(-\beta t^2) \tag{5.1.9}$$

$$\begin{aligned} e^{\left(\frac{xt}{\alpha}\right)} e^{(-\beta t^2)} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{xt}{\alpha}\right)^n}{n!} \frac{(-\beta\ t^2)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{x}{\alpha}\right)^n}{n!} \frac{(-\beta)^m}{m!} t^{n+2m} \end{aligned} \tag{5.1.10}$$

Using series identity (1..22.5) (replacing $n \longrightarrow n-2m$)

$$e^{\left(\frac{xt}{\alpha}\right)} e^{(-\beta t^2)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\left(\frac{x}{\alpha}\right)^{n-2m}}{(n-2m)!} \frac{(-\beta)^m}{m!} t^n \tag{5.1.11}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{\left(\frac{x}{\alpha}\right)^n}{(1)_n} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\left(\frac{x}{\alpha}\right)^{-2m}}{(1+n)_{-2m}} \frac{(-\beta)^m}{m!} \right\} t^n \tag{5.1.12}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ \left(\frac{x}{\alpha}\right)^n \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{2^{2m} \left(\frac{-n}{2}\right)_m \left(\frac{-n+1}{2}\right)_m \left(\frac{-\alpha^2\beta}{x^2}\right)^m}{m!} \right\} \frac{t^n}{n!} \tag{5.1.13}$$

Comparing the coefficient of t^n both side, we get

$$H_{n,\alpha,\beta}(x) = \left(\frac{x}{\alpha}\right)^n {}_2F_0\left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}; \; \frac{-4\alpha^2\beta}{x^2} \\ - \quad ; \end{array}\right] \tag{5.1.14}$$

Put $\alpha = \frac{1}{2}, \beta = 1$ we get Classical Hermite polynomial $H_n(x)$.

5.1.2 Hypergeometric Form of Lagrange's Polynomial in Two Variables

The familiar (classical two-variable) polynomials $g_n^{(\alpha,\beta)}(x,y)$ generated by

$$(1-xt)^{-\alpha}(1-yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n \tag{5.1.15}$$

$$(|t| < min\{|x|^{-1}, |y|^{-1}\})$$

are known as the Lagrange polynomials (1867) in two variables which occur in certain problems in statistics [95, p. 267 (1)].

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = (1-xt)^{-\alpha}(1-yt)^{-\beta} \tag{5.1.16}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = {}_1F_0\left[\begin{array}{c} \alpha; \; xt \\ -; \end{array}\right] {}_1F_0\left[\begin{array}{c} \beta; \; yt \\ -; \end{array}\right] \tag{5.1.17}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n t^n}{n!} \sum_{m=0}^{\infty} \frac{(\beta)_m y^m t^m}{m!} \tag{5.1.18}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_n x^n}{n!} \frac{(\beta)_m y^m t^{n+m}}{m!} \tag{5.1.19}$$

Using series identity (1.22.1) (replacing $n \longrightarrow n-m$), we have

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\alpha)_{n-m} x^{n-m}}{(n-m)!} \frac{(\beta)_m y^m t^n}{(m)!} \tag{5.1.20}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\alpha)_n(\alpha+n)_{-m} x^n}{n!} \frac{(\beta)_m(-n)_m}{(-1)^m m!} \left(\frac{y}{x}\right)^m t^n \tag{5.1.21}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\left[\frac{x^n(\alpha)_n}{n!} \sum_{m=0}^{n} \frac{(-1)^m(\beta)_m(-n)_m}{(-1)^m m!(1-\alpha-n)_m} \left(\frac{y^m}{x^m}\right)\right] t^n \tag{5.1.22}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty} \frac{x^n(\alpha)_n}{n!} {}_2F_1 \left[\begin{array}{cc} -n, & \beta; \\ 1-\alpha-n\ ; \end{array} \frac{y}{x} \right] t^n \tag{5.1.23}$$

$$g_n^{(\alpha,\beta)}(x,y) = \frac{x^n(\alpha)_n}{n!} {}_2F_1 \left[\begin{array}{cc} -n, & \beta; \\ 1-\alpha-n\ ; \end{array} \frac{y}{x} \right] \tag{5.1.24}$$

Again

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = (1-xt)^{-\alpha}(1-yt)^{-\beta} \tag{5.1.25}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = {}_1F_0 \left[\begin{array}{c} \beta; \\ -; \end{array} yt \right] {}_1F_0 \left[\begin{array}{c} \alpha; \\ -; \end{array} xt \right] \tag{5.1.26}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_m x^m t^m}{m!} \frac{(\beta)_n y^n t^n}{n!} \tag{5.1.27}$$

Using series identity (1.22.1) (replacing $n \longrightarrow n-m$), we have

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\alpha)_m x^m}{m!} \frac{(\beta)_{n-m} y^{n-m} t^n}{(n-m)!} \tag{5.1.28}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\alpha)_m x^m}{m!} \frac{(\beta)_n(\beta+n)_{-m} y^n y^{-m} t^n (-n)_m}{n!(-1)^m} \tag{5.1.29}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty} \left[\frac{y^n(\beta)_n}{n!} \sum_{m=0}^{n} \frac{(\alpha)_m(-n)_m(-1)^m}{(-1)^m m!(1-\beta-n)_m} \left(\frac{x^m}{y^m}\right) \right] t^n \tag{5.1.30}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty} \frac{y^n(\beta)_n}{n!} {}_2F_1 \left[\begin{array}{cc} -n, & \alpha; \\ 1-\beta-n\ ; \end{array} \frac{x}{y} \right] t^n \tag{5.1.31}$$

$$g_n^{(\alpha,\beta)}(x,y) = \frac{y^n(\beta)_n}{n!} {}_2F_1 \left[\begin{array}{cc} -n, & \alpha; \\ 1-\beta-n\ ; \end{array} \frac{x}{y} \right] \tag{5.1.32}$$

5.2 Linear Generating Relations and Hypergeometric Transformations Associated with Dickinson-Warsi Polynomials

Any values of parameters and variables leading to the results given in this section which do not make sense, are tacitly excluded.

$$\sum_{n=0}^{\infty} H_{2n+2m}^{(a-n)}(x)\frac{t^n}{n!} = \frac{(1-4t)^{(a-\frac{1}{2})}}{2x} exp[4x^2t] H_{2m+1}^{(a-1)}\left[x\sqrt{(1-4t)}\ \right] \tag{5.2.1}$$

$$\sum_{n=0}^{\infty} H_{2n+2m+1}^{(a-n)}(x)\frac{t^n}{n!} = 2x(1-4t)^{(a+1)}exp[4x^2t]H_{2m}^{(a+1)}\left[x\sqrt{(1-4t)}\ \right] \quad (5.2.2)$$

$$\sum_{n=0}^{\infty} H_{2n+2m}^{(a)}(x)\frac{t^n}{n!} = \frac{(1+4t)^{-(a+m+\frac{1}{2})}}{2x}exp\left[\frac{4x^2t}{1+4t}\right]H_{2m+1}^{(a-1)}\left[\frac{x}{\sqrt{(1+4t)}}\ \right] \quad (5.2.3)$$

$$\sum_{n=0}^{\infty} H_{2n+2m+1}^{(a)}(x)\frac{t^n}{n!} = 2x\,(1+4t)^{-(a+m+2)}\,exp\left[\frac{4x^2t}{1+4t}\right]H_{2m}^{(a+1)}\left[\frac{x}{\sqrt{(1+4t)}}\ \right] \quad (5.2.4)$$

$$ {}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2}; \\ \quad \frac{1}{2} \quad ; \end{array} \frac{4x^2y}{(1+y)^2}\right]$$

$$= (1+y)^a\,{}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2}; \\ \quad \frac{1}{2} \quad ; \end{array} \left(x+\sqrt{(x^2-1)}\ \right)^2 y\right]{}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2}; \\ \quad \frac{1}{2} \quad ; \end{array} \left(x-\sqrt{(x^2-1)}\ \right)^2 y\right]$$

$$+y\,(1+y)^a\,a^2{}_2F_1\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2}; \\ \quad \frac{3}{2} \quad ; \end{array} \left(x+\sqrt{(x^2-1)}\right)^2 y\right]{}_2F_1\left[\begin{array}{l} \frac{a+1}{2},\frac{a+2}{2}; \\ \quad \frac{3}{2} \quad ; \end{array} \left(x-\sqrt{(x^2-1)}\ \right)^2 y\right] \quad (5.2.5)$$

$$ {}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2}; \\ \quad \frac{3}{2} \quad ; \end{array} \frac{4x^2y}{(1+y)^2}\right] = \frac{(1+y)^a\left(x+\sqrt{(x^2-1)}\ \right)}{2x}{}_2F_1\left[\begin{array}{l} \frac{a-1}{2},\frac{a}{2}; \\ \quad \frac{1}{2} \quad ; \end{array} \left(x-\sqrt{(x^2-1)}\ \right)^2 y\right]\times$$

$$\times{}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2}; \\ \quad \frac{3}{2} \quad ; \end{array} \left(x+\sqrt{(x^2-1)}\ \right)^2 y\right] + \frac{(1+y)^a\left(x-\sqrt{(x^2-1)}\ \right)}{2x}\times$$

$$\times\,{}_2F_1\left[\begin{array}{l} \frac{a-1}{2},\frac{a}{2}; \\ \quad \frac{1}{2} \quad ; \end{array} \left(x+\sqrt{(x^2-1)}\right)^2 y\right]{}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{a+1}{2}; \\ \quad \frac{3}{2} \quad ; \end{array} \left(x-\sqrt{(x^2-1)}\ \right)^2 y\right] \quad (5.2.6)$$

Provided that $\left|\frac{x^2y}{(1+y)^2}\right| < \frac{1}{4}$.

5.3 Linear Generating Relations Involving Subuhi Khan Polynomials

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{t^n}{(2n)!} = \cosh\left(\frac{x\sqrt{t}}{\alpha}\right)\left({}_0F_1\left[\begin{array}{l} -\ ; \\ \frac{1}{2}\ ; \end{array} \frac{\beta^2t^2}{4}\right] - \beta t\ {}_0F_1\left[\begin{array}{l} -\ ; \\ \frac{3}{2}\ ; \end{array} \frac{\beta^2t^2}{4}\right]\right) \quad (5.3.1)$$

$$\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{t^n}{(2n+1)!} = \frac{1}{\sqrt{t}}\sinh\left(\frac{x\sqrt{t}}{\alpha}\right)\left({}_0F_1\left[\begin{array}{l} -\ ; \\ \frac{1}{2}\ ; \end{array} \frac{\beta^2t^2}{4}\right] - \beta t\ {}_0F_1\left[\begin{array}{l} -\ ; \\ \frac{3}{2}\ ; \end{array} \frac{\beta^2t^2}{4}\right]\right) \quad (5.3.2)$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta)}{n!} = exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\cos\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right) \tag{5.3.3}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\sin(n\theta)}{n!} = exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\sin\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right) \tag{5.3.4}$$

5.4 Linear Generating Relations Associated with Lagrange's Polynomials in Two Variables

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\cos\ n\theta = \frac{1}{\left(\sqrt{(1+x^2-2x\cos\theta)}\right)^{\alpha}\left(\sqrt{(1+y^2-2y\cos\theta)}\right)^{\beta}}$$

$$\cos\left(\alpha\tan^{-1}\left(\frac{x\sin\theta}{x\cos\theta-1}\right)+\beta\tan^{-1}\left(\frac{y\sin\theta}{y\cos\theta-1}\right)\right) \tag{5.4.1}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\sin\ n\theta = -\frac{1}{\left(\sqrt{(1+x^2-2x\cos\theta)}\right)^{\alpha}\left(\sqrt{(1+y^2-2y\cos\theta)}\right)^{\beta}}$$

$$\sin\left(\alpha\tan^{-1}\left(\frac{x\sin\theta}{x\cos\theta-1}\right)+\beta\tan^{-1}\left(\frac{y\sin\theta}{y\cos\theta-1}\right)\right) \tag{5.4.2}$$

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)t^n$$

$$= {}_2F_1\left[\begin{array}{ll}\frac{\alpha}{2},\frac{\alpha+1}{2}; & x^2t\\ \frac{1}{2}\quad ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\frac{\beta}{2},\frac{\beta+1}{2}; & y^2t\\ \frac{1}{2}\quad ; & \end{array}\right]+\alpha\,\beta\,x\,y\,t\,{}_2F_1\left[\begin{array}{ll}\frac{\alpha+1}{2},\frac{\alpha+2}{2}; & x^2t\\ \frac{3}{2}\quad ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\frac{\beta+1}{2},\frac{\beta+2}{2}; & y^2t\\ \frac{3}{2}\quad ; & \end{array}\right] \tag{5.4.3}$$

$$\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)t^n$$

$$= \beta y\,{}_2F_1\left[\begin{array}{ll}\frac{\alpha}{2},\frac{\alpha+1}{2}; & x^2t\\ \frac{1}{2}\quad ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\frac{\beta+1}{2},\frac{\beta+2}{2}; & y^2t\\ \frac{3}{2}\quad ; & \end{array}\right]+\alpha x\,{}_2F_1\left[\begin{array}{ll}\frac{\alpha+1}{2},\frac{\alpha+2}{2}; & x^2t\\ \frac{3}{2}\quad ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\frac{\beta}{2},\frac{\beta+1}{2}; & y^2t\\ \frac{1}{2}\quad ; & \end{array}\right] \tag{5.4.4}$$

5.5 Derivation of all Generating Relations and Transformations

Consider the series

$$S_1=\sum_{n=0}^{\infty} H_{2n+2m}^{(a-n)}(\sqrt{x})\frac{t^n}{n!} \tag{5.5.1}$$

Using the result(5.1.5), we get

$$S_1=(-1)^m 2^{2m}\sum_{n=0}^{\infty}(m+n)!\ L_{m+n}^{(a-n)}(x)\frac{(-4t)^n}{n!} \tag{5.5.2}$$

Now using the result (5.1.8) in (5.5.2), we get

$$S_1=(-1)^m 2^{2m} m!(1-4t)^a exp[4xt]L_m^{(a)}[x(1-4t)] \tag{5.5.3}$$

Now applying (5.1.6) and replacing x by x^2, we get the generating relation (5.2.1).

If we apply same technique for the series

$$S_2=\sum_{n=0}^{\infty} H_{2n+2m+1}^{(a-n)}\left(\sqrt{x}\right)\frac{t^n}{n!} \tag{5.5.4}$$

we get the generating relation (5.2.2).

Now consider the another series

$$S_3=\sum_{n=0}^{\infty} H_{2n+2m}^{(a)}\left(\sqrt{x}\right)\frac{t^n}{n!} \tag{5.5.5}$$

Using the results (5.1.5) and (5.1.7) in equation (5.5.5), we get

$$S_3=(-1)^m 2^{2m} m!(1+4t)^{-1-a-m} exp\left[\frac{4xt}{1+4t}\right] L_m^{(a)}\left[\frac{x}{1+4t}\right]$$

Now using the result (5.1.6) and replacing x by x^2, we get the generating relation (5.2.3).

Similarly we obtain the generating relation (5.2.4) from the following series

$$S_4=\sum_{n=0}^{\infty} H_{2n+2m+1}^{(a)}(\sqrt{x})\frac{t^n}{n!} \tag{5.5.6}$$

Now consider the following identity

$$\left(1+y^2\right)^{-a}\left[1-\frac{2xy}{1+y^2}\right]^{-a}=\left[1-\left(x+\sqrt{x^2-1}\right)y\right]^{-a}\left[1-\left(x-\sqrt{x^2-1}\right)y\right]^{-a}$$

Writing $\left[1-\dfrac{2xy}{1+y^2}\right]^{-a}$, $\left[1-(x+\sqrt{x^2-1})y\right]^{-a}$ and $\left[1-(x-\sqrt{x^2-1})y\right]^{-a}$ in hypergeometric notations, using Barr's identity (5.1.2), replacing y by $i\sqrt{y}$, equating real and imaginary parts and making suitable adjustment of parameters, we get the hypergeometric transformations (5.2.5) and (5.2.6). For similar method, readers are advised to refer the paper [52].

Consider the generating relation given by Subuhi [170]

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = exp\left(\frac{xt}{\alpha}\right) exp(-\beta t^2) \tag{5.5.7}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = exp\left(\frac{xt}{\alpha}\right) \sum_{r=o}^{\infty} \frac{(-\beta t^2)^r}{r!} \tag{5.5.8}$$

Apply Decomposition technique

$$\sum_{m=0}^{\infty} \Phi(m) = \sum_{m=0}^{\infty} \Phi(2m) + \sum_{m=0}^{\infty} \Phi(2m+1) \tag{5.5.9}$$

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{t^{2n}}{(2n)!}+\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{t^{2n+1}}{(2n+1)!} = exp\left(\frac{xt}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(-\beta t^2)^{2r}}{(2r)!}+\sum_{r=o}^{\infty}\frac{(-\beta t^2)^{2r+1}}{(2r+1)!}\right] \tag{5.5.10}$$

Put $t = iT$, or $t^2 = -T^2$, we get

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n)!} + iT\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n+1)!}$$

$$= exp\left(\frac{ixT}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(\beta T^2)^{2r}}{(2r)!}+\sum_{r=o}^{\infty}\frac{(\beta T^2)^{2r+1}}{(2r+1)!}\right] \tag{5.5.11}$$

Equating Real and imaginary parts

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n)!} = \cos\left(\frac{xT}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{1}{2})_r}+(\beta T^2)\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{3}{2})_r}\right] \tag{5.5.12}$$

$$\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n+1)!} = \frac{1}{T}\sin\left(\frac{xT}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{1}{2})_r}+(\beta T^2)\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{3}{2})_r}\right] \tag{5.5.13}$$

Put $T = i\sqrt{t}$ or $T^2 = -t$, we get

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{t^n}{(2n)!} = \cosh\left(\frac{x\sqrt{t}}{\alpha}\right)\left({}_0F_1\left[\begin{matrix} - \, ; & \frac{\beta^2t^2}{4} \\ \frac{1}{2}\, ; & \end{matrix}\right] - \beta t\ {}_0F_1\left[\begin{matrix} - \, ; & \frac{\beta^2t^2}{4} \\ \frac{3}{2}\, ; & \end{matrix}\right]\right) \tag{5.5.14}$$

$$\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{t^n}{(2n+1)!} = \frac{1}{\sqrt{t}}\sinh\left(\frac{x\sqrt{t}}{\alpha}\right)\left({}_0F_1\left[\begin{matrix} - ; & \frac{\beta^2t^2}{4} \\ \frac{1}{2} ; & \end{matrix}\right] - \beta t\ {}_0F_1\left[\begin{matrix} - ; & \frac{\beta^2t^2}{4} \\ \frac{3}{2} ; & \end{matrix}\right]\right) \tag{5.5.15}$$

Another method for Decomposition technique is given by

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = exp\left(\frac{xt}{\alpha}\right) exp(-\beta t^2) \tag{5.5.16}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = exp\left(\frac{xt}{\alpha} - \beta t^2\right) \tag{5.5.17}$$

Put $t = \cos\theta + i\sin\theta = e^{i\theta}$ in both side and apply Demoivre's theorem, we get

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta) + i\sin(n\theta)}{n!} = exp\left(\frac{x(\cos\theta + i\sin\theta)}{\alpha} - \beta\{\cos(2\theta) + i\sin(2\theta)\}\right) \tag{5.5.18}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta)}{n!} + i\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\sin(n\theta)}{n!}$$
$$= exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right) exp\ i\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right) \tag{5.5.19}$$

Equating real and imaginary parts

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta)}{n!} = exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\cos\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right) \tag{5.5.20}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\sin(n\theta)}{n!} = exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\sin\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right) \tag{5.5.21}$$

Consider the generating relations for Lagrange's polynomial of two variable

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = (1-xt)^{-\alpha}(1-yt)^{-\beta} \tag{5.5.22}$$

Apply Decomposition technique, put $t = \cos\theta + i\sin\theta$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)(\cos\theta + i\sin\theta)^n = \{1-x(\cos\theta+i\sin\theta)\}^{-\alpha}\{1-y(\cos\theta+i\sin\theta)\}^{-\beta} \tag{5.5.23}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)(\cos\ n\theta + i\sin\ n\theta) = \{(1-x\cos\theta) - ix\sin\theta\}^{-\alpha}\{(1-y\cos\theta) - iy\sin\theta\}^{-\beta} \tag{5.5.24}$$

Put $1 - x\cos\theta = r\cos\phi$, $-x\sin\theta = r\sin\phi$

Put $1 - y\cos\theta = R\cos\Psi$, $-y\sin\theta = R\sin\Psi$

on solving

$$r^2 = 1 + x^2 - 2x\cos\theta$$

$$\tan\phi = \frac{-x\sin\theta}{1-x\cos\theta}$$

similarly

$$R^2 = 1 + y^2 - 2y\cos\theta$$

$$\tan\Psi = \frac{-y\sin\theta}{1-y\cos\theta}$$

Put these values in above equation

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)(\cos\ n\theta + i\sin\ n\theta) = r^{-\alpha}R^{-\beta}\{\cos(\alpha\phi+\beta\Psi) - i\sin(\alpha\phi+\beta\Psi)\} \tag{5.5.25}$$

Equating real and imaginary parts

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\cos\ n\theta = \frac{1}{r^{\alpha}R^{\beta}}\cos(\alpha\phi+\beta\Psi) \tag{5.5.26}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\sin\ n\theta = -\frac{1}{r^{\alpha}R^{\beta}}\sin(\alpha\phi+\beta\Psi) \tag{5.5.27}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\cos\ n\theta = \frac{1}{\left(\sqrt{(1+x^2-2x\cos\theta)}\right)^{\alpha}\left(\sqrt{(1+y^2-2y\cos\theta)}\right)^{\beta}}$$

$$\cos\left(\alpha\tan^{-1}\left(\frac{x\sin\theta}{x\cos\theta-1}\right)+\beta\tan^{-1}\left(\frac{y\sin\theta}{y\cos\theta-1}\right)\right) \tag{5.5.28}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\sin\ n\theta = -\frac{1}{\left(\sqrt{(1+x^2-2x\cos\theta)}\right)^{\alpha}\left(\sqrt{(1+y^2-2y\cos\theta)}\right)^{\beta}}$$

$$\sin\left(\alpha\tan^{-1}\left(\frac{x\sin\theta}{x\cos\theta-1}\right)+\beta\tan^{-1}\left(\frac{y\sin\theta}{y\cos\theta-1}\right)\right) \tag{5.5.29}$$

Another method for Decomposition technique

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = (1-xt)^{-\alpha}(1-yt)^{-\beta} \tag{5.5.30}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = {}_1F_0\left[\begin{matrix}\alpha; \\ -;\end{matrix}\ xt\right]{}_1F_0\left[\begin{matrix}\beta; \\ -;\end{matrix}\ yt\right] \tag{5.5.31}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{p=0}^{\infty}(\alpha)_p\frac{(xt)^p}{p!}\sum_{q=0}^{\infty}(\beta)_q\frac{(yt)^q}{q!} \tag{5.5.32}$$

Now applying identity (5.5.9)

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)t^{2n} + \sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)t^{2n+1}$$

$$= \left(\sum_{p=0}^{\infty}(\alpha)_{2p}\frac{(xt)^{2p}}{(2p)!} + \sum_{p=0}^{\infty}(\alpha)_{2p+1}\frac{(xt)^{2p+1}}{(2p+1)!}\right)\left(\sum_{q=0}^{\infty}(\beta)_{2q}\frac{(yt)^{2q}}{(2q)!} + \sum_{q=0}^{\infty}(\beta)_{2q+1}\frac{(yt)^{2q+1}}{(2q+1)!}\right) \tag{5.5.33}$$

Put $t = iT$ or $t^2 = -T^2$, we have

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n + iT\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= \left(\sum_{p=0}^{\infty}(\alpha)_{2p}\frac{x^{2p}(-T^2)^p}{(2p)!} + iT\sum_{p=0}^{\infty}(\alpha)_{2p+1}\frac{x^{2p+1}(-T^2)^p}{(2p+1)!}\right)\times$$

$$\times\left(\sum_{q=0}^{\infty}(\beta)_{2q}\frac{y^{2q}(-T^2)^q}{(2q)!} + iT\sum_{q=0}^{\infty}(\beta)_{2q+1}\frac{y^{2q+1}(-T^2)^q}{(2q+1)!}\right) \tag{5.5.34}$$

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n + iT\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= \left(\sum_{p=0}^{\infty}\frac{2^{2p}\left(\frac{\alpha}{2}\right)_p\left(\frac{\alpha+1}{2}\right)_p}{2^{2p}\left(\frac{1}{2}\right)_p}\frac{x^{2p}(-T^2)^p}{p!} + iT\alpha x\sum_{p=0}^{\infty}\frac{2^{2p}\left(\frac{\alpha+1}{2}\right)_p\left(\frac{\alpha+2}{2}\right)_p}{2^{2p}\left(\frac{3}{2}\right)_p}\frac{x^{2p}(-T^2)^p}{p!}\right)\times$$

$$\times\left(\sum_{q=0}^{\infty}\frac{2^{2q}\left(\frac{\beta}{2}\right)_q\left(\frac{\beta+1}{2}\right)_q}{2^{2q}\left(\frac{1}{2}\right)_q}\frac{y^{2q}(-T^2)^q}{q!} + iT\beta y\sum_{q=0}^{\infty}\frac{2^{2q}\left(\frac{\beta+1}{2}\right)_q\left(\frac{\beta+2}{2}\right)_q}{2^{2q}\left(\frac{3}{2}\right)_q}\frac{y^{2q}(-T^2)^q}{q!}\right) \tag{5.5.35}$$

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n + iT\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= \left({}_2F_1\left[\begin{array}{l}\frac{\alpha}{2}, \frac{\alpha+1}{2};\\ \frac{1}{2}\quad ;\end{array} -x^2T^2\right] + iT\alpha x\,{}_2F_1\left[\begin{array}{l}\frac{\alpha+1}{2}, \frac{\alpha+2}{2};\\ \frac{3}{2}\quad ;\end{array} -x^2T^2\right]\right)\times$$

$$\times\left({}_2F_1\left[\begin{array}{l}\frac{\beta}{2}, \frac{\beta+1}{2};\\ \frac{1}{2}\quad ;\end{array} -y^2T^2\right] + iT\beta y\,{}_2F_1\left[\begin{array}{l}\frac{\beta+1}{2}, \frac{\beta+2}{2};\\ \frac{3}{2}\quad ;\end{array} -y^2T^2\right]\right) \tag{5.5.36}$$

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n + iT\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= {}_2F_1\left[\begin{array}{l}\frac{\alpha}{2}, \frac{\alpha+1}{2};\\ \frac{1}{2}\quad ;\end{array} -x^2T^2\right]{}_2F_1\left[\begin{array}{l}\frac{\beta}{2}, \frac{\beta+1}{2};\\ \frac{1}{2}\quad ;\end{array} -y^2T^2\right]$$

$$-\alpha\beta xy\,T^2{}_2F_1\left[\begin{array}{l}\frac{\alpha+1}{2}, \frac{\alpha+2}{2};\\ \frac{3}{2}\quad ;\end{array} -x^2T^2\right]{}_2F_1\left[\begin{array}{l}\frac{\beta+1}{2}, \frac{\beta+2}{2};\\ \frac{3}{2}\quad ;\end{array} -y^2T^2\right]$$

$$+iT\beta y\,{}_2F_1\left[\begin{array}{l}\frac{\alpha}{2}, \frac{\alpha+1}{2};\\ \frac{1}{2}\quad ;\end{array} -x^2T^2\right]{}_2F_1\left[\begin{array}{l}\frac{\beta+1}{2}, \frac{\beta+2}{2};\\ \frac{3}{2}\quad ;\end{array} -y^2T^2\right]$$

$$+ iT\alpha x {}_2F_1\left[\begin{matrix}\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} - x^2T^2\right] {}_2F_1\left[\begin{matrix}\frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} - y^2T^2\right] \tag{5.5.37}$$

Equating real and imaginary parts, we have

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= {}_2F_1\left[\begin{matrix}\frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} - x^2T^2\right] {}_2F_1\left[\begin{matrix}\frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} - y^2T^2\right]$$

$$- \alpha\beta xyT^2 {}_2F_1\left[\begin{matrix}\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} - x^2T^2\right] {}_2F_1\left[\begin{matrix}\frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} - y^2T^2\right] \tag{5.5.38}$$

$$\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= \beta y {}_2F_1\left[\begin{matrix}\frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} - x^2T^2\right] {}_2F_1\left[\begin{matrix}\frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} - y^2T^2\right]$$

$$+ \alpha x {}_2F_1\left[\begin{matrix}\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} - x^2T^2\right] {}_2F_1\left[\begin{matrix}\frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} - y^2T^2\right] \tag{5.5.39}$$

Put $T = i\sqrt{t}$ or $T^2 = -t$, we have

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)t^n$$

$$= {}_2F_1\left[\begin{matrix}\frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} x^2t\right] {}_2F_1\left[\begin{matrix}\frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} y^2t\right] + \alpha\,\beta\,x\,y\,t\,{}_2F_1\left[\begin{matrix}\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} x^2t\right] {}_2F_1\left[\begin{matrix}\frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} y^2t\right] \tag{5.5.40}$$

$$\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)t^n$$

$$= \beta y\,{}_2F_1\left[\begin{matrix}\frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} x^2t\right] {}_2F_1\left[\begin{matrix}\frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} y^2t\right] + \alpha x\,{}_2F_1\left[\begin{matrix}\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ;\end{matrix} x^2t\right] {}_2F_1\left[\begin{matrix}\frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ;\end{matrix} y^2t\right] \tag{5.5.41}$$

Chapter 6

Multiple Generating Relations Involving Double and Triple Hypergeometric Functions

6.1 Introduction

The name generating function was introduced by Laplace in 1812. Since then the theory of generating functions has been developed into various directions and found wide applications in various branches of science and technology.

Most of the generating functions derived are the extensions and generalizations of the results known in one form or another in the theory of special function. There is a vast literature on generating functions.

A generating functions may be used to define a set of functions, to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, et-cetera.

In this chapter we derive two Multiple generating relations involving Exton's double hypergeometric function, and Kampé de Fériet's function.

Dixon theorem [252, p.92 Art(53)]

$$ {}_3F_2\left[\begin{matrix} A, & B, & C; \\ 1+A-B, & 1+A-C; & \end{matrix} 1\right] = \frac{\Gamma(1+\frac{A}{2})\Gamma(1+A-B)\Gamma(1+A-C)\Gamma(1+\frac{A}{2}-B-C)}{\Gamma(1+A)\Gamma(1+\frac{A}{2}-B)\Gamma(1+\frac{A}{2}-C)\Gamma(1+A-B-C)} \tag{6.1.1} $$

where $\Re(\frac{A}{2}-B-C) > -1$.

For convenience, we shall use the notation $\Delta(N;\lambda)$ for array of N parameters given by $\frac{\lambda}{N}, \frac{\lambda+1}{N}, \frac{\lambda+2}{N}, \ldots, \frac{\lambda+N-1}{N}$.

Thus the symbol $\Delta[N;(a_p)]$ denotes the array of parameters given by $\Delta(N;a_1), \Delta(N;a_2), \Delta(N;a_3), \ldots, \Delta(N;a_p)$ with similar interpretations for others.

Thus the symbol $\Delta[N;1-(a_p)-n]$ denotes the array of parameters given by $\Delta(N;1-a_1-n), \Delta(N;1-a_2-n), \Delta(N;1-a_3-n), \ldots, \Delta(N;1-a_p-n)$, with similar interpretations for others.

The symbol $\Delta[N;1-(a_p)-n]$ denotes the array of pN parameters $\frac{1-a_j-n+j}{N}$; $i=1,2,3,\ldots,p$; $j=0,1,2,\ldots,N-1$, with similar interpretations for others.

6.2 Expansion of t^n in a Series of Generalized Hypergeometric Polynomials

To prove

$$t^m = \frac{m!\,[(k_K)]_m}{[(a_A)]_m} \sum_{r=0}^{m} \frac{(-1)^r (b+2r)(b)_r}{r!(m-r)!(b)_{m+r+1}} \; {}_{2+A}F_K \left[\begin{array}{c} -r, b+r, (a_A); \\ (k_K) \quad ; \end{array} t \right] \quad (6.2.1)$$

Proof: Suppose right hand side is denoted by Ω

$$\Omega = \frac{m!\,[(k_K)]_m}{[(a_A)]_m} \sum_{r=0}^{m} \frac{(-1)^r (b+2r)(b)_r}{r!(m-r)!(b)_{m+r+1}} \; {}_{2+A}F_K \left[\begin{array}{c} -r, b+r, (a_A); \\ (k_K) \quad ; \end{array} t \right] \quad (6.2.2)$$

$$= \frac{m!\,[(k_K)]_m}{[(a_A)]_m} \sum_{r=0}^{m} \frac{(-1)^r (b)_{2r+1}(b)_r}{r!(b)_{2r}(m-r)!(b)_{m+r+1}} \sum_{s=0}^{r} \frac{(-r)_s (b+r)_s\,[(a_A)]_s}{[(k_K)]_s} \frac{t^s}{s!}$$

$$= \frac{b\,[(k_K)]_m}{[(a_A)]_m\,(b)_{m+1}} \sum_{r=0}^{m} \sum_{s=0}^{r} \frac{(b+1)_{2r}(b)_{r+s}}{(b)_{2r}(b+m+1)_r} \frac{(-1)^s\;(-m)_r\,[(a_A)]_s}{(r-s)!\;[(k_K)]_s} \frac{t^s}{s!}$$

Using Double summation identity (1.22.9)

$$\sum_{r=0}^{m} \sum_{s=0}^{r} B(s,r) = \sum_{r=0}^{m} \sum_{s=0}^{m-r} B(s,s+r) \quad (6.2.3)$$

We have

$$\Omega = \frac{b\,[(k_K)]_m}{[(a_A)]_m\,(b)_{m+1}} \sum_{r=0}^{m} \sum_{s=0}^{m-r} \frac{(b+1)_{2(r+s)}(b)_{r+2s}}{(b)_{2(r+s)}(b+m+1)_{r+s}} \frac{(-1)^s\;(-m)_{r+s}\,[(a_A)]_s}{(r)!\;[(k_K)]_s} \frac{t^s}{s!} \quad (6.2.4)$$

$$= \frac{b\,[(k_K)]_m}{[(a_A)]_m\,(b)_{m+1}} \sum_{r,s=0}^{r+s\le m} \frac{(b+1)_{2(r+s)}(b)_{r+2s}\;(-m)_{r+s}}{(b)_{2(r+s)}(r)!\;(b+m+1)_{r+s}} \frac{(-1)^s\,[(a_A)]_s}{[(k_K)]_s} \frac{t^s}{s!} \quad (6.2.5)$$

$$= \frac{b\,[(k_K)]_m}{[(a_A)]_m\,(b)_{m+1}} \sum_{s=0}^{m} \frac{(b+1)_{2s}(b)_{2s}\;(-m)_s(-1)^s\,[(a_A)]_s}{(b)_{2s}\;(b+m+1)_s\,[(k_K)]_s} \frac{t^s}{s!} \sum_{r=0}^{m-s} \frac{(b+1+2s)_{2r}(-m+s)_r(b+2s)_r}{(b+2s)_{2r}(r)!\;(b+m+1+s)_r} \quad (6.2.6)$$

$$= \frac{b\,[(k_K)]_m}{[(a_A)]_m\,(b)_{m+1}} \sum_{s=0}^{m} \frac{(b+1)_{2s}\;(-m)_s(-1)^s\,[(a_A)]_s}{(b+m+1)_s\,[(k_K)]_s} \frac{t^s}{s!} \sum_{r=0}^{m-s} \frac{(-m+s)_r \left(\frac{b}{2}+1+s\right)_r (b+2s)_r}{\left(\frac{b}{2}+s\right)_r (r)!\;(b+m+1+s)_r} \quad (6.2.7)$$

$$= \frac{b\,[(k_K)]_m}{[(a_A)]_m\,(b)_{m+1}} \sum_{s=0}^{m} \frac{(b+1)_{2s}\;(-m)_s(-1)^s\,[(a_A)]_s}{(b+m+1)_s\,[(k_K)]_s} \frac{t^s}{s!} \; {}_3F_2 \left[\begin{array}{c} b+2s, -m+s, \frac{b}{2}+1+s; \\ 1+b+s+m, \quad \frac{b}{2}+s; \end{array} 1 \right] \quad (6.2.8)$$

Using equation (6.1.1), we have

$$\Omega=\frac{b\left[(k_K)\right]_m}{\left[(a_A)\right]_m(b)_{m+1}}\sum_{s=0}^{m}\frac{(b+1)_{2s}\;(-m)_s(-1)^s\left[(a_A)\right]_s}{(b+m+1)_s\left[(k_K)\right]_s}\frac{t^s}{s!}\frac{\Gamma(1+\frac{b}{2}+s)\Gamma(1+b+s+m)}{\Gamma(1+b+2s)\Gamma(1+\frac{b}{2}+s+m-s)}\times$$

$$\times\frac{\Gamma(\frac{b}{2}+s)\Gamma(1+\frac{b}{2}+s+m-s-\frac{b}{2}-1-s)}{\Gamma(1+\frac{b}{2}+s-1-\frac{b}{2}-s)\Gamma(1+b+2s+m-s-\frac{b}{2}-1-s)}\tag{6.2.9}$$

When $s=0,1,2,3,4,\ldots,m-1$, corresponding ${}_3F_2$ function will be zero and when $s=m$ then ${}_3F_2=1$

$$\begin{aligned}\Omega&=\frac{b\left[(k_K)\right]_m}{\left[(a_A)\right]_m(b)_{m+1}}\frac{(b+1)_{2m}\;(-m)_m(-1)^m\left[(a_A)\right]_m}{(b+m+1)_m\left[(k_K)\right]_m}\frac{t^m}{m!}\\&=\frac{b}{(b)_{m+1}}\frac{(b+1)_{2m}\;(-1)^m\;m!(-1)^m}{(b+m+1)_m}\frac{t^m}{m!}\\&=t^m=LHS\end{aligned}\tag{6.2.10}$$

6.3 Multiple Generating Relations

$$(1-xt)^{-a}F^{B+n:E;H}_{D\;\;:G;L}\left[\begin{array}{l}\Delta(n;a),(b_B):(e_E);(h_H);\\\quad(d_D)\qquad\;:(g_G);(\ell_L)\;;\end{array}\frac{y}{(1-xt)^n}\frac{z}{(1-xt)^n}\right]$$

$$=\sum_{j=0}^{n-1}\sum_{r=0}^{\infty}\frac{(a)_{j+r}x^{j+r}\left[(k_K)\right]_{j+r}}{\left[(a_A)\right]_{j+r}}\frac{(-1)^r(b+2r)(b)_r}{(r)!(j)!(b)_{j+2r+1}}{}_{2+A}F_K\left[\begin{array}{l}-r,b+r,(a_A);\\\quad(k_K)\quad;\end{array}t\right]\times$$

$$\times F^{(3)}\left[\begin{array}{l}\Delta(n;a+j+r)::-;(b_B);-:\qquad 1,\\\qquad-\qquad::-;(d_D);-:\Delta[n;(a_A)+j+r],\Delta(n;1+j),\end{array}\right.$$

$$\left.\begin{array}{l}\Delta[n;(k_K)+j+r]\quad;(e_E);(h_H);\\\Delta(n;b+j+2r+1);(g_G);(\ell_L)\;;\end{array}\frac{x^n}{n^{n(A-K+1)}},y,z\right]\tag{6.3.1}$$

$$(1-xt)^{-a}\mathscr{H}^{A+n:0;D;G}_{B\;\;\;:0;E;H}\left[\begin{array}{l}\Delta(n;a),(a_A):-;(d_D);(g_G);\\(b_B)\qquad\quad:-;(e_E);(h_H);\end{array}\frac{y}{(1-xt)^{2n}}\frac{z}{(1-xt)^{n}}\right]$$

$$=\sum_{i=0}^{1}\sum_{j=0}^{2n-1}\sum_{r=0}^{\infty}\frac{(a)_{r+j+ni}(-x)^r\,[(k_K)]_{r+j}\,[(a_A)]_i\,[(g_G)]_i\,x^j}{r!\;i!\;j!\,[(\ell_L)]_{r+j}\,[(b_B)]_i\,[(h_H)]_i\,(b+r)_r\;(b+2r+1)_j}\left(\frac{z}{n^n}\right)^i\times$$

$$\times{}_{2+L}F_K\left[\begin{array}{l}-r,b+r,(\ell_L);\\\quad(k_K)\qquad;\end{array}t\right]F^{(3)}\left[\begin{array}{l}\Delta(2n;a+r+ni+j)::-;\Delta[2;(a_A)+i];-:\\\qquad-\qquad\qquad::-;\Delta[2;(b_B)+i];-:\end{array}\right.$$

$$\begin{array}{l}1,\qquad\qquad\Delta[2n;(k_K)+j+r]\qquad;(d_D);\\\Delta(2n;1+j),\Delta[2n;(\ell_L)+j+r],\Delta(2n;b+2r+1+j);(e_E);\end{array}$$

$$\left.\begin{array}{l}1,\qquad\Delta[2;(g_G)+i]\;;\\\Delta(2;1+i),\Delta[2;(h_H)+i];\end{array}(2n)^{2n(K-L-1)}\;x^{2n},\;4^{(A-B+n)}\;y,\;4^{(A-B+n-1+G-H)}\;z^2\right]\tag{6.3.2}$$

6.4 Derivation of Generating Relations

Derivation of Generating Relation I:

$$(1-xt)^{-a}F^{B+n:E;H}_{D\;\;\;:G;L}\left[\begin{array}{l}\Delta(n;a),(b_B):(e_E);(h_H);\\\quad(d_D)\qquad:(g_G);(\ell_L)\;;\end{array}\frac{y}{(1-xt)^{n}}\frac{z}{(1-xt)^{n}}\right]$$

$$=\sum_{p,q=0}^{\infty}\frac{\prod_{j=1}^{n}\left(\frac{a+j-1}{n}\right)_{p+q}[(b_B)]_{p+q}\,[(e_E)]_p\,[(h_H)]_q}{[(d_D)]_{p+q}\,[(g_G)]_p\,[(\ell_L)]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}(1-xt)^{-(a+np+nq)}\tag{6.4.1}$$

$$=\sum_{p,q=0}^{\infty}\frac{(a)_{np+nq}\,[(b_B)]_{p+q}\,[(e_E)]_p\,[(h_H)]_q}{n^{np+nq}\,[(d_D)]_{p+q}\,[(g_G)]_p\,[(\ell_L)]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\sum_{m=0}^{\infty}\frac{(a+np+nq)_m}{m!}x^mt^m\tag{6.4.2}$$

$$=\sum_{m,p,q=0}^{\infty}\frac{(a)_{np+nq+m}\,[(b_B)]_{p+q}\,[(e_E)]_p\,[(h_H)]_q}{n^{np+nq}\,[(d_D)]_{p+q}\,[(g_G)]_p\,[(\ell_L)]_q}\frac{x^m}{(m)!}\frac{y^p}{(p)!}\frac{z^q}{(q)!}t^m\tag{6.4.3}$$

Using equation (6.2.1), we have

$$(1-xt)^{-a}F^{B+n:E;H}_{D\ :G;L}\left[\begin{array}{l}\Delta(n;a),(b_B):(e_E);(h_H);\\ (d_D)\qquad\quad :(g_G);(\ell_L)\ ;\end{array}\frac{y}{(1-xt)^n}\frac{z}{(1-xt)^n}\right]$$

$$=\sum_{m,p,q=0}^{\infty}\frac{(a)_{np+nq+m}\left[(b_B)\right]_{p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}{n^{np+nq}\left[(d_D)\right]_{p+q}\left[(g_G)\right]_p\left[(\ell_L)\right]_q}\frac{x^m}{(m)!}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\times$$

$$\times\frac{m!\left[(k_K)\right]_m}{\left[(a_A)\right]_m}\sum_{r=0}^{m}\frac{(-1)^r(b+2r)(b)_r}{r!(m-r)!(b)_{m+r+1}}{}_{2+A}F_K\left[\begin{array}{l}-r,b+r,(a_A);\\ (k_K)\qquad ;\end{array} t\right] \tag{6.4.4}$$

Using equation (1.22.2)

$$(1-xt)^{-a}F^{B+n:E;H}_{D\ :G;L}\left[\begin{array}{l}\Delta(n;a),(b_B):(e_E);(h_H);\\ (d_D)\qquad\quad :(g_G);(\ell_L)\ ;\end{array}\frac{y}{(1-xt)^n}\frac{z}{(1-xt)^n}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{p,q=0}^{\infty}\frac{(a)_{np+nq+m+r}\left[(b_B)\right]_{p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}{n^{np+nq}\left[(d_D)\right]_{p+q}\left[(g_G)\right]_p\left[(\ell_L)\right]_q}\frac{x^{m+r}}{(m+r)!}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\times$$

$$\times\frac{(m+r)!\left[(k_K)\right]_{m+r}}{\left[(a_A)\right]_{m+r}}\frac{(-1)^r(b+2r)(b)_r}{r!(m)!(b)_{m+2r+1}}{}_{2+A}F_K\left[\begin{array}{l}-r,b+r,(a_A);\\ (k_K)\qquad ;\end{array} t\right] \tag{6.4.5}$$

Using series identity (1.22.11)

$$\sum_{m=0}^{\infty}\Psi(m)=\sum_{j=0}^{n-1}\sum_{m=0}^{\infty}\Psi(mn+j) \tag{6.4.6}$$

where $\forall n\in\{1,2,3,\ldots\}$ We have

$$(1-xt)^{-a}F^{B+n:E;H}_{D\ :G;L}\left[\begin{array}{l}\Delta(n;a),(b_B):(e_E);(h_H);\\ (d_D)\qquad\quad :(g_G);(\ell_L)\ ;\end{array}\frac{y}{(1-xt)^n}\frac{z}{(1-xt)^n}\right]$$

$$=\sum_{j=0}^{n-1}\sum_{m=0}^{\infty}\sum_{r,p,q=0}^{\infty}\frac{(a)_{np+nq+nm+j+r}\left[(b_B)\right]_{p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}{n^{np+nq}\left[(d_D)\right]_{p+q}\left[(g_G)\right]_p\left[(\ell_L)\right]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\times$$

$$\times\frac{x^{nm+j+r}\left[(k_K)\right]_{nm+j+r}}{\left[(a_A)\right]_{nm+j+r}}\frac{(-1)^r(b+2r)(b)_r}{r!(nm+j)!(b)_{nm+j+2r+1}}{}_{2+A}F_K\left[\begin{array}{l}-r,b+r,(a_A);\\ (k_K)\qquad ;\end{array} t\right] \tag{6.4.7}$$

$$=\sum_{j=0}^{n-1}\sum_{r=0}^{\infty}\frac{(a)_{j+r}x^{j+r}\left[(k_K)\right]_{j+r}}{\left[(a_A)\right]_{j+r}}\frac{(-1)^r(b+2r)(b)_r}{r!(1)_j(b)_{j+2r+1}}{}_{2+A}F_K\left[\begin{array}{l}-r,b+r,(a_A);\\ (k_K)\qquad ;\end{array} t\right]\times$$

$$\times\sum_{m,p,q=0}^{\infty}\frac{(a+j+r)_{n(p+q+m)}\left[(b_B)\right]_{p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}{n^{np+nq}\left[(d_D)\right]_{p+q}\left[(g_G)\right]_p\left[(\ell_L)\right]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\times$$

$$\times\frac{(x^n)^m\left[(k_K)+j+r\right]_{nm}}{(1+j)_{nm}(b+j+2r+1)_{nm}\left[(a_A)+j+r\right]_{nm}} \tag{6.4.8}$$

Using equation (1.2.17), we have

$$(1-xt)^{-a}F^{B+n:E;H}_{D\ :G;L}\left[\begin{array}{ll}\Delta(n;a),(b_B):(e_E);(h_H); & \dfrac{y}{(1-xt)^n}\dfrac{z}{(1-xt)^n}\\ (d_D)\qquad :(g_G);(\ell_L)\ ; & \end{array}\right]$$

$$=\sum_{j=0}^{n-1}\sum_{r=0}^{\infty}\frac{(a)_{j+r}x^{j+r}\left[(k_K)\right]_{j+r}}{\left[(a_A)\right]_{j+r}}\frac{(-1)^r(b+2r)(b)_r}{(r)!(j)!(b)_{j+2r+1}}{}_{2+A}F_K\left[\begin{array}{ll}-r,b+r,(a_A); & t\\ (k_K)\quad ; & \end{array}\right]\times$$

$$\times\sum_{m,p,q=0}^{\infty}\frac{\prod_{i=1}^{n}\left(\frac{a+j+r+i-1}{n}\right)_{m+p+q}\left[(b_B)\right]_{p+q}}{m!\ p!\ q!\left[(d_D)\right]_{p+q}\prod_{i=1}^{n}\left(\frac{1+j+i-1}{n}\right)_m}.\frac{(1)_m\prod_{i=1}^{n}\left(\frac{(k_K)+j+r+i-1}{n}\right)_m}{\prod_{i=1}^{n}\left(\frac{(a_A)+j+r+i-1}{n}\right)_m}\times$$

$$\times\frac{\left[(e_E)\right]_p\left[(h_H)\right]_q}{\prod_{i=1}^{n}\left(\frac{b+j+2r+1+i-1}{n}\right)_m\left[(g_G)\right]_p\left[(\ell_L)\right]_q}y^pz^q\left(\frac{x^n}{n^{-(nK)+nA+n}}\right)^m \qquad (6.4.9)$$

$$(1-xt)^{-a}F^{B+n:E;H}_{D\ :G;L}\left[\begin{array}{ll}\Delta(n;a),(b_B):(e_E);(h_H); & \dfrac{y}{(1-xt)^n}\dfrac{z}{(1-xt)^n}\\ (d_D)\qquad :(g_G);(\ell_L)\ ; & \end{array}\right]$$

$$=\sum_{j=0}^{n-1}\sum_{r=0}^{\infty}\frac{(a)_{j+r}x^{j+r}\left[(k_K)\right]_{j+r}}{\left[(a_A)\right]_{j+r}}\frac{(-1)^r(b+2r)(b)_r}{(r)!(j)!(b)_{j+2r+1}}{}_{2+A}F_K\left[\begin{array}{ll}-r,b+r,(a_A); & t\\ (k_K)\quad ; & \end{array}\right]\times$$

$$\times F^{(3)}\left[\begin{array}{ll}\Delta(n;a+j+r)::-;(b_B);-: & 1,\qquad \Delta[n;(k_K)+j+r]\quad ;\\ -\quad ::-;(d_D);-: & \Delta[n;(a_A)+j+r],\Delta(n;1+j),\Delta(n;b+j+2r+1);\end{array}\right.$$

$$\left.\begin{array}{ll}(e_E);(h_H); & \dfrac{x^n}{n^{n(A-K+1)}},y,z\\ (g_G);(\ell_L)\ ; & \end{array}\right] \qquad (6.4.10)$$

Derivation of Generating Relation II:

$$(1-xt)^{-a}\mathscr{H}^{A+n:0;D;G}_{B\ :0;E;H}\left[\begin{array}{ll}\Delta(n;a),(a_A):-;(d_D);(g_G); & \dfrac{y}{(1-xt)^{2n}}\dfrac{z}{(1-xt)^n}\\ (b_B)\qquad :-;(e_E);(h_H); & \end{array}\right]$$

$$=\sum_{p,q=0}^{\infty}\frac{\prod_{j=1}^{n}\left(\frac{a+j-1}{n}\right)_{2p+q}\left[(a_A)\right]_{2p+q}\left[(d_D)\right]_p\left[(g_G)\right]_q}{\left[(b_B)\right]_{2p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\ {}_1F_0\left[\begin{array}{ll}a+2np+nq; & xt\\ -\quad ; & \end{array}\right] \qquad (6.4.11)$$

$$=\sum_{p,q=0}^{\infty}\frac{(a)_{2np+nq}\left[(a_A)\right]_{2p+q}\left[(d_D)\right]_p\left[(g_G)\right]_q}{n^{2np+nq}\left[(b_B)\right]_{2p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\sum_{m=0}^{\infty}\frac{(a+2np+nq)_m}{m!}x^mt^m \qquad (6.4.12)$$

$$= \sum_{p,q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n(2p+q)+m}\left[(a_A)\right]_{2p+q}\left[(d_D)\right]_p\left[(g_G)\right]_q}{n^{n(2p+q)}\left[(b_B)\right]_{2p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q} \frac{y^p}{(p)!}\frac{z^q}{(q)!}\frac{x^m}{(m)!}t^m \tag{6.4.13}$$

Using equation (6.2.1), we have

$$(1-xt)^{-a}\mathscr{H}_{B\quad :0;E;H}^{A+n:0;D;G}\left[\begin{array}{lc} \Delta(n;a),(a_A) : -;(d_D);(g_G); & \frac{y}{(1-xt)^{2n}}\frac{z}{(1-xt)^n} \\ (b_B) \qquad\quad : -;(e_E);(h_H); & \end{array}\right]$$

$$= \sum_{p,q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n(2p+q)+m}\left[(a_A)\right]_{2p+q}\left[(d_D)\right]_p\left[(g_G)\right]_q}{n^{n(2p+q)}\left[(b_B)\right]_{2p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q} \frac{y^p}{(p)!}\frac{z^q}{(q)!}\frac{x^m}{(m)!}\times$$

$$\times \frac{m!\left[(k_K)\right]_m}{\left[(\ell_L)\right]_m}\sum_{r=0}^{m}\frac{(-1)^r(b+2r)(b)_r}{r!(m-r)!(b)_{m+r+1}}\,{}_{2+L}F_K\left[\begin{array}{c} -r,b+r,(\ell_L); \\ (k_K) \quad ; \end{array} t\right] \tag{6.4.14}$$

Using equation (1.22.2)

$$(1-xt)^{-a}\mathscr{H}_{B\quad :0;E;H}^{A+n:0;D;G}\left[\begin{array}{lc} \Delta(n;a),(a_A) : -;(d_D);(g_G); & \frac{y}{(1-xt)^{2n}}\frac{z}{(1-xt)^n} \\ (b_B) \qquad\quad : -;(e_E);(h_H); & \end{array}\right]$$

$$= \sum_{p,q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_{n(2p+q)+m+r}\left[(a_A)\right]_{2p+q}\left[(d_D)\right]_p\left[(g_G)\right]_q}{n^{n(2p+q)}\left[(b_B)\right]_{2p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q} \frac{y^p}{(p)!}\frac{z^q}{(q)!}\frac{x^{(m+r)}}{(m)!}\times$$

$$\times \frac{\left[(k_K)\right]_{(m+r)}}{\left[(\ell_L)\right]_{(m+r)}}\frac{(-1)^r(b+2r)(b)_r}{r!(b)_{m+2r+1}}\,{}_{2+L}F_K\left[\begin{array}{c} -r,b+r,(\ell_L); \\ (k_K) \quad ; \end{array} t\right] \tag{6.4.15}$$

$$= \sum_{r=0}^{\infty}\frac{x^r(a)_r\left[(k_K)\right]_r}{\left[(\ell_L)\right]_r}\frac{(-1)^r}{r!(b+r)_r}\,{}_{2+L}F_K\left[\begin{array}{c} -r,b+r,(\ell_L); \\ (k_K) \quad ; \end{array} t\right]\times$$

$$\times \sum_{m,p,q=0}^{\infty}\frac{(a+r)_{n(2p+q)+m}\left[(k_K)+r\right]_m\left[(a_A)\right]_{2p+q}\left[(d_D)\right]_p\left[(g_G)\right]_q}{n^{n(2p+q)}\left[(\ell_L)+r\right]_m(b+2r+1)_m\left[(b_B)\right]_{2p+q}\left[(e_E)\right]_p\left[(h_H)\right]_q}\frac{y^p}{(p)!}\frac{z^q}{(q)!}\frac{x^m}{(m)!} \tag{6.4.16}$$

Using series identity (1.22.12), We have

$$(1-xt)^{-a}\mathscr{H}_{B\quad :0;E;H}^{A+n:0;D;G}\left[\begin{array}{lc} \Delta(n;a),(a_A) : -;(d_D);(g_G); & \frac{y}{(1-xt)^{2n}}\frac{z}{(1-xt)^n} \\ (b_B) \qquad\quad : -;(e_E);(h_H); & \end{array}\right]$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{2n-1}\sum_{r=0}^{\infty}\frac{x^r(a)_r\left[(k_K)\right]_r}{\left[(\ell_L)\right]_r}\frac{(-1)^r}{r!(b+r)_r}\,{}_{2+L}F_K\left[\begin{array}{c} -r,b+r,(\ell_L); \\ (k_K) \quad ; \end{array} t\right]\times$$

$$\times \sum_{m,p,q=0}^{\infty}\frac{(a+r)_{2np+2nq+2mn+j+ni}\left[(a_A)\right]_{2p+2q+i}\left[(d_D)\right]_p\left[(g_G)\right]_{2q+i}\left[(k_K)+r\right]_{2mn+j}}{n^{n(2p+2q+i)}\left[(\ell_L)+r\right]_{2mn+j}(b+2r+1)_{2mn+j}\left[(b_B)\right]_{2p+2q+i}\left[(e_E)\right]_p\left[(h_H)\right]_{2q+i}}\times$$

$$\times \frac{y^p}{(p)!}\frac{z^{2q+i}}{(2q+i)!}\frac{x^{2mn+j}}{(2mn+j)!} \tag{6.4.17}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{2n-1}\sum_{r=0}^{\infty} \frac{x^r (a)_r \left[(k_K)\right]_r}{\left[(\ell_L)\right]_r} \frac{(-1)^r}{r!(b+r)_r} {}_{2+L}F_K \left[\begin{array}{c} -r, b+r, (\ell_L); \\ (k_K) \quad ; \end{array} t \right] \times$$

$$\times \frac{(a+r)_{j+ni} \left[(a_A)\right]_i \left[(g_G)\right]_i \left[(k_K)+r\right]_j z^i \; x^j}{(1)_j (1)_i n^{ni} \left[(\ell_L)+r\right]_j (b+2r+1)_j \left[(b_B)\right]_i \left[(h_H)\right]_i} \times$$

$$\times \sum_{m,p,q=0}^{\infty} \frac{(a+r+j+ni)_{2n(m+p+q)} \left[(a_A)+i\right]_{2(p+q)} \left[(d_D)\right]_p \left[(g_G)+i\right]_{2q} \left[(k_K)+r+j\right]_{2mn}}{n^{2n(p+q)} \left[(\ell_L)+r+j\right]_{2mn} (b+2r+1+j)_{2mn} \left[(b_B)+i\right]_{2(p+q)} \left[(e_E)\right]_p \left[(h_H)+i\right]_{2q}} \times$$

$$\times \frac{y^p}{(p)!} \frac{z^{2q}}{(1+i)_{2q}} \frac{x^{2mn}}{(1+j)_{2mn}} \tag{6.4.18}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{2n-1}\sum_{r=0}^{\infty} \frac{x^{r+j} (a)_{r+j+ni} (-1)^r \left[(k_K)\right]_{r+j} \left[(a_A)\right]_i \left[(g_G)\right]_i}{\left[(\ell_L)\right]_{r+j} r!\; i!\; j! \left[(b_B)\right]_i \left[(h_H)\right]_i (b+r)_r \;\; (b+2r+1)_j} \left(\frac{z}{n^n}\right)^i \times$$

$$\times {}_{2+L}F_K \left[\begin{array}{c} -r, b+r, (\ell_L); \\ (k_K) \quad ; \end{array} t \right] \sum_{m,p,q=0}^{\infty} \frac{(a+r+j+ni)_{2n(m+p+q)} \left[(a_A)+i\right]_{2(p+q)}}{n^{2n(p+q)} \left[(\ell_L)+r+j\right]_{2mn} (b+2r+1+j)_{2mn}} \times$$

$$\times \frac{\left[(d_D)\right]_p \left[(g_G)+i\right]_{2q} \left[(k_K)+r+j\right]_{2mn}}{\left[(b_B)+i\right]_{2(p+q)} \left[(e_E)\right]_p \left[(h_H)+i\right]_{2q}} \frac{y^p}{(p)!} \frac{z^{2q}}{(1+i)_{2q}} \frac{x^{2mn}}{(1+j)_{2mn}} \tag{6.4.19}$$

Using equation (1.2.17)

$$(1-xt)^{-a} \mathscr{H}_{B\;\;:0;E;H}^{A+n:0;D;G} \left[\begin{array}{c} \Delta(n;a), (a_A) : -; (d_D); (g_G); \\ (b_B) \quad : -; (e_E); (h_H); \end{array} \frac{y}{(1-xt)^{2n}} \frac{z}{(1-xt)^n} \right]$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{2n-1}\sum_{r=0}^{\infty} \frac{(a)_{r+j+ni} (-x)^r \left[(k_K)\right]_{r+j} \left[(a_A)\right]_i \left[(g_G)\right]_i x^j}{r!\; i!\; j! \left[(\ell_L)\right]_{r+j} \left[(b_B)\right]_i \left[(h_H)\right]_i (b+r)_r \;\; (b+2r+1)_j} \left(\frac{z}{n^n}\right)^i \times$$

$$\times {}_{2+L}F_K \left[\begin{array}{c} -r, b+r, (\ell_L); \\ (k_K) \quad ; \end{array} t \right] \sum_{m,p,q=0}^{\infty} \frac{\prod_{\ell=1}^{2n} \left(\frac{a+j+r+ni+\ell-1}{2n} \right)_{m+p+q}}{m!\; p!\; q! \prod_{\ell=1}^{2} \left(\frac{(b_B)+i+\ell-1}{2} \right)_{p+q}} \times$$

$$\times \frac{\prod_{\ell=1}^{2} \left(\frac{(a_A)+i+\ell-1}{2} \right)_{p+q} (1)_m \prod_{\ell=1}^{2n} \left(\frac{(k_K)+r+j+\ell-1}{2n} \right)_m}{\prod_{\ell=1}^{2n} \left(\frac{1+j+\ell-1}{2n} \right)_m \prod_{s=1}^{2n} \left(\frac{(\ell_L)+r+j+s-1}{2n} \right)_m \prod_{s=1}^{2n} \left(\frac{b+2r+1+j+s-1}{2n} \right)_m} \times$$

$$\times \frac{\left[(d_D)\right]_p (1)_q \prod_{\ell=1}^{2} \left(\frac{(g_G)+i+\ell-1}{2} \right)_q}{\left[(e_E)\right]_p \prod_{\ell=1}^{2} \left(\frac{1+i+\ell-1}{2} \right)_q \prod_{\ell=1}^{2} \left(\frac{(h_H)+i+\ell-1}{2} \right)_q} \times$$

$$\times (x^{2n})^m (2n)^{2nm(K-L-1)} \; 4^{(A-B+n)p} y^p \; 4^{(A+G+n-1-B-H)q} (z^2)^q \tag{6.4.20}$$

$$(1-xt)^{-a}\mathscr{H}_{B\ :0;E;H}^{A+n:0;D;G}\left[\begin{array}{c} \Delta(n;a),(a_A):-;(d_D);(g_G); \\ (b_B)\qquad :-;(e_E);(h_H); \end{array} \frac{y}{(1-xt)^{2n}}\frac{z}{(1-xt)^{n}}\right]$$

$$=\sum_{i=0}^{1}\sum_{j=0}^{2n-1}\sum_{r=0}^{\infty}\frac{(a)_{r+j+ni}(-x)^r\,[(k_K)]_{r+j}\,[(a_A)]_i\,[(g_G)]_i\,x^j}{r!\ i!\ j!\,[(\ell_L)]_{r+j}\,[(b_B)]_i\,[(h_H)]_i\,(b+r)_r\ (b+2r+1)_j}\left(\frac{z}{n^n}\right)^i\times$$

$$\times{}_{2+L}F_K\left[\begin{array}{c} -r,b+r,(\ell_L); \\ (k_K)\qquad ; \end{array} t\right]F^{(3)}\left[\begin{array}{c} \Delta(2n;a+r+ni+j)::-;\Delta[2;(a_A)+i];-: \\ -\qquad ::-;\Delta[2;(b_B)+i];-: \end{array}\right.$$

$$\begin{array}{c} 1,\qquad \Delta[2n;(k_K)+j+r]\qquad ;(d_D); \\ \Delta(2n;1+j),\Delta[2n;(\ell_L)+j+r],\Delta(2n;b+2r+1+j);(e_E); \end{array}$$

$$\left.\begin{array}{c} 1,\qquad \Delta[2;(g_G)+i]\ ; \\ \Delta(2;1+i),\Delta[2;(h_H)+i]; \end{array}(2n)^{2n(K-L-1)}\ x^{2n},\ 4^{(A-B+n)}\ y,\ 4^{(A-B+n-1+G-H)}\ z^2\right] \tag{6.4.21}$$

Chapter 7

Finite Double Sums

7.1 Introduction

One of the most useful techniques to obtain finite sums of hypergeometric functions and their generalizations consists in the elementary manipulations of series. Some Mathematician had derived some useful as well as interesting generalizations from Vandermonde's summation theorem. Besides, there are a large number of other finite sums of hypergeometric functions in the literature.

In this chapter we obtain four finite double sums of Kamp*é* de F*é*riets double hypergeometric function of higher order using Gausss summation theorem. Associated and contiguous relations for Kummers confluent hypergeometric function, Gausss ordinary hypergeometric function, Saalsch*ü*tzs summation theorem, double sums involving Appell, Humbert, Karlsson, Bessel functions, are obtained as special cases of double summations.

Any values of parameters and variables leading to the results given in sections 2 and 4 which do not make sense, are tacitly excluded.

7.2 Four Finite Double Summations

In this section we obtain the following four finite double summation formulae with the help of series manipulations.

If the values of a and b are adjusted in such a way that $(1-b)$, $(1+a-b)$ are not integers; c is neither zero nor a negative integer then

$$\sum_{r=0}^{n}\sum_{s=0}^{m} C(m,n,r,s,a,b,c) F^{P+1:D;E}_{Q\ \ :G;H+1}\left[\begin{matrix}(p_P), a+r : (d_D); (e_E) \quad ; \\ (q_Q) \quad : (g_G); (h_H), c+n+s;\end{matrix}\ x, y\right]$$

$$= \frac{(b)_{m+n}}{(c)_{m+n}} F^{P+2:D;E+1}_{Q+1:G;H+2}\left[\begin{matrix}(p_P), a, b+n : (d_D); \quad (e_E), b+m+n \quad ; \\ (q_Q), b \quad : (g_G); (h_H), b+n, c+m+n;\end{matrix}\ x, y\right] \tag{7.2.1}$$

$$\sum_{r=0}^{n}\sum_{s=0}^{m} C(m,n,r,s,a,b,c) F^{P\ \ :D+1;E}_{Q+1:\ G\ \ ;H}\left[\begin{matrix}(p_P) \quad : (d_D), a+r; (e_E); \\ (q_Q), c+n+s : \quad (g_G) \quad ; (h_H);\end{matrix}\ x, y\right]$$

$$= \frac{(b)_{m+n}}{(c)_{m+n}} F^{P+1:D+2;E}_{Q+2:G+1;H}\left[\begin{array}{l} (p_P), b+m+n \quad : (d_D), a, b+n; (e_E); \\ (q_Q), b+n, c+m+n : \quad (g_G), b \quad ; (h_H); \end{array} x, y\right] \tag{7.2.2}$$

$$\sum_{r=0}^{n}\sum_{s=0}^{m} C(m,n,r,s,a,b,c) F^{P:D+1;E}_{Q:\ G\ ;H+1}\left[\begin{array}{l} (p_P) : (d_D), a+r;\ (e_E) \quad ; \\ (q_Q) : \quad (g_G) \quad ; (h_H), c+n+s; \end{array} x, y\right]$$

$$= \frac{(b)_{m+n}}{(c)_{m+n}} F^{P:D+2;E+1}_{Q:G+1;H+2}\left[\begin{array}{l} (p_P) : (d_D), a, b+n; (e_E), b+m+n \quad ; \\ (q_Q) : (g_G), b \quad ; (h_H), b+n, c+m+n; \end{array} x, y\right] \tag{7.2.3}$$

$$\sum_{r=0}^{n}\sum_{s=0}^{m} C(m,n,r,s,a,b,c) F^{P+1:D;E}_{Q+1:G;H}\left[\begin{array}{l} (p_P), a+r \quad : (d_D); (e_E); \\ (q_Q), c+n+s : (g_G); (h_H); \end{array} x, y\right]$$

$$= \frac{(b)_{m+n}}{(c)_{m+n}} F^{P+2:D;E}_{Q+2:G;H}\left[\begin{array}{l} (p_P), b+m+n, a : (d_D); (e_E); \\ (q_Q), c+m+n, b : (g_G); (h_H); \end{array} x, y\right] \tag{7.2.4}$$

where

$$C(m,n,r,s,a,b,c) = \binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r(c-b)_s}{(a-b+1)_{r-n}(c)_{n+s}} \tag{7.2.5}$$

and Pochhammer symbols and Gamma functions are well defined; denominator parameters in hypergeometric notations are neither zero nor negative integers.

7.3 Derivation of Finite Double Summations

On expressing each Kampé de Fériet's double hypergeometric function of left hand sides of (7.2.1), (7.2.2), (7.2.3) and (7.2.4) in power series forms with the help of (1.10.1) and (7.2.5), interchanging the order of summations with the help of (1.2.12), (1.2.13) and (1.2.14) and successive applications of Gauss's summation theorem (1.3.6), we get the right hand sides of (7.2.1), (7.2.2), (7.2.3) and (7.2.4).

7.4 Applications

Making suitable adjustment of parameters in (7.2.1), (7.2.2), (7.2.3) and (7.2.4) we can find a number of finite double sums involving double hypergeometric functions of Appell and Humbert [91, pp.224-226].

Putting $P = E = 1$, $Q = D = G = H = 0$ in (7.2.2) and using a result of Carlson [37, p.222 (4)], we have following double sums

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r(c-b)_s}{(a-b+1)_{r-n}(c)_{n+s}}F^{2:1;0}_{1:1;0}\left[\begin{array}{l} p, a+r+e : a+r \quad ;-; \\ c+n+s \quad : a+r+e ;-; \end{array} x-y, y\right]$$

$$= \frac{(b)_{m+n}}{(c)_{m+n}}F^{2:2;1}_{2:1;0}\left[\begin{array}{l} p, b+m+n \quad : a, b+n; e \,; \\ b+n, c+m+n \,: b \quad ;-; \end{array} x, y\right] \tag{7.4.1}$$

When $y = x$ and $e = 0$, (7.4.1) reduces to a known result of Pathan [216, p.58 (2.1)] which was obtained with the help of operational calculus technique

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r(c-b)_s}{(a-b+1)_{r-n}(c)_{n+s}}\,{}_2F_1\left[\begin{array}{c} a+r, p \;; \\ c+n+s; \end{array} x\right]$$

$$= \frac{(b)_{m+n}}{(c)_{m+n}}\,{}_3F_2\left[\begin{array}{c} a, b+m+n, p; \\ b, c+m+n \quad ; \end{array} x\right] \tag{7.4.2}$$

where ${}_2F_1$ and ${}_3F_2$ are Gauss's ordinary hypergeometric function [252, p.45 (1)] and Clausen's hypergeometric function [252, p.73 (2)], respectively.

In (7.4.2) putting $n = 0$,$m = 1$,$b = a$ and $c = d - 1$, we get a known contiguous relation [252, p.71 Ex.21(2,13)]

$$(a-d+1)\,{}_2F_1\left[\begin{array}{c} a, p \;; \\ d \;\; ; \end{array} x\right] = a\;{}_2F_1\left[\begin{array}{c} a+1, p \;; \\ d \;\; ; \end{array} x\right] - (d-1)\;{}_2F_1\left[\begin{array}{c} a, p \;; \\ d-1; \end{array} x\right] \tag{7.4.3}$$

When $m = 0$, $x = 1$ and $p = b$ in (7.4.2) and using Gauss's summation theorem (1.3.6), we get Saalschütz's summation theorem[252, p.87 (Th. 29)] in the following form

$${}_3F_2\left[\begin{array}{c} -n, \quad a, \quad 1+a-c-n \;; \\ 1+a-b-n, 1+a+b-c-n; \end{array} 1\right] = \frac{(c-b)_n(b)_n}{(b-a)_n(c-a-b)_n} \tag{7.4.4}$$

because sum of its denominator parameters exceeds the sum of its numerator parameters by unity.

Putting $P = Q = H = D = 0$, $G = E = 1$ and setting $e_1 = b + n - 1$, $g_1 = a$, $c = 1 + b$ in (7.2.3), and using a result of Karlsson [144, p.197 (8)] for ${}_2F_2$, we get a result involving the product of two Kummer's confluent hypergeometric functions ${}_1F_1$ [252, p.123 (1)]

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r s!}{(a-b+1)_{r-n}(1+b)_{n+s}}\,{}_1F_1\left[\begin{array}{c}a+r;\\ a\ \ ;\end{array} x\right]{}_1F_1\left[\begin{array}{c}b+n-1\ \ ;\\ 1+b+n+s;\end{array} y\right]$$

$$=\frac{b}{m+1}\,{}_1F_1\left[\begin{array}{c}b+n;\\ b\ \ ;\end{array} x\right]{}_1F_1\left[\begin{array}{c}b+n-1;\\ b+n\ \ ;\end{array} y\right]+\frac{b(1-b-n)}{(m+1)(b+m+n)}\times$$

$$\times\,{}_1F_1\left[\begin{array}{c}b+n;\\ b\ \ ;\end{array} x\right]{}_1F_1\left[\begin{array}{c}b+m+n\ \ ;\\ 1+b+m+n;\end{array} y\right] \tag{7.4.5}$$

When $P=Q=E=H=D=G=0$ in (7.2.3) and $y=-\dfrac{z^2}{4}$ we get

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r(c-b)_s}{(a-b+1)_{r-n}(1-x)^r}\left(\frac{2}{z}\right)^s J_{c+n+s-1}(z)$$

$$=(1-x)^a\left(\frac{z}{2}\right)^{c+n-1}\frac{(b)_{m+n}}{\Gamma(c+m+n)}\,{}_2F_1\left[\begin{array}{c}a,b+n;\\ b\ \ ;\end{array} x\right]{}_1F_2\left[\begin{array}{c}b+m+n\ \ ;\\ b+n,c+m+n;\end{array} -\frac{z^2}{4}\right] \tag{7.4.6}$$

where $J_m(z)$ is Bessel function of first kind of order m [252, p.108 (1)].

When $m=0$, (7.4.2) reduces to

$$\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^{n+r}(a)_r}{(a-b+1)_{r-n}}\,{}_2F_1\left[\begin{array}{c}a+r,p\ ;\\ c+n\ \ ;\end{array} x\right]=(b)_n\ {}_3F_2\left[\begin{array}{c}a,b+n,p\ ;\\ b,\ c+n\ \ ;\end{array} x\right] \tag{7.4.7}$$

In (7.4.5), replacing x and y by xt and yt respectively, multiplying both the sides by $e^{-t}t^{c-1}$ and integrating term by term with respect to t from 0 to ∞, we have

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r s!}{(a-b+1)_{r-n}(1+b)_{n+s}}F_2[c;a+r,b+n-1;a,1+b+n+s;x,y]$$

$$=\frac{b(1-b-n)}{(m+1)(b+m+n)}F_2[c;b+n,b+m+n;b,1+b+m+n;x,y]$$

$$+\frac{b}{m+1}F_2[c;b+n,b+n-1;b,b+n;x,y] \tag{7.4.8}$$

where F_2 is Appell's double hypergeometric function of second kind [91, p.224 (7)].

In (7.4.8), replacing x and y by xt and $y(1-t)$ respectively, multiplying both the sides by $t^{k-1}(1-t)^{c-k-1}$ and integrating term by term with respect to t from 0 to 1, we have

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r s!}{(a-b+1)_{r-n}(1+b)_{n+s}}\,{}_2F_1\left[\begin{array}{c}k, a+r;\\ a\quad ;\end{array} x\right]{}_2F_1\left[\begin{array}{c}c-k, b+n-1;\\ 1+b+n+s\quad ;\end{array} y\right]$$

$$= \frac{b}{m+1}\,{}_2F_1\left[\begin{array}{c}k, b+n;\\ b\quad ;\end{array} x\right]{}_2F_1\left[\begin{array}{c}c-k, b+n-1;\\ b+n\quad ;\end{array} y\right] + \frac{b(1-b-n)}{(m+1)(b+m+n)}\times$$

$$\times\,{}_2F_1\left[\begin{array}{c}k, b+n;\\ b\quad ;\end{array} x\right]{}_2F_1\left[\begin{array}{c}c-k, b+m+n;\\ 1+b+m+n\quad ;\end{array} y\right] \tag{7.4.9}$$

When $P = E = 1$, $G = D = Q = H = 0$ and $p_1 = p_2 + p_3$ in (7.2.2) and using a reduction formula of Karlsson [146, p.40 (4.5)], we get

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\binom{n}{r}\binom{m}{s}\frac{(-1)^{n+r+s}(a)_r(c-b)_s}{(a-b+1)_{r-n}(c)_{n+s}}H_c^{(4)}\left[a+r, p_2, e, p_3; c+n+s; x, y, y, x\right]$$

$$= \frac{(b)_{m+n}}{(c)_{n+m}}F^{2:2;1}_{2:1;0}\left[\begin{array}{c}p_2+p_3, b+m+n : a, b+n; e\ ;\\ b+n, c+m+n\quad :\quad b\quad ; -;\end{array} x, y\right] \tag{7.4.10}$$

In all equations the values of a and b are adjusted in such a way that $(1-b)$, $(1+a-b)$ are not integers; c is neither zero nor a negative integer; Pochhammer symbols and Gamma functions are well defined and denominator parameters in hypergeometric notations are neither zero nor negative integers.

Chapter 8

Laplace Transforms of Some Functions Using Mellin-Barnes Type Contour Integration

8.1 Introduction

In this chapter, we find some results on Laplace transforms of elementary, composite functions, Special functions and Multiple hypergeometric functions like $\arcsin t$, $\arctan t$, $\cos(b \arcsin t)$, square of inverse hyperbolic and trigonometric functions, logarithmic functions and other algebraic functions, products of Ordinary Bessel's function of I kind and Modified Bessel's function of I kind, five Complete Elliptic integrals, Hyper-Bessel function of Humbert, Modified Hyper-Bessel function of Delerue, Error function, Incomplete Beta function, Incomplete Gamma function, Fresnel Integrals, Sine Integral, Hyperbolic Sine Integral, Polylogarithm function, Struve function, Modified Struve function, Lommel function, Kelvin's functions and Lerch's Transcendent, product theorems of Bailey, Cayley, Clausen, Orr, Preece, Watson for Gauss functions ${}_2F_1$, Appell's four functions, Ramanujan's product theorems, Henrici's triple product theorem, Whipple's quadratic transformation, Gauss, Goursat quadratic and cubic transformations, Product theorems of Kummer's functions ${}_1F_1$, Product theorems of Bessel's functions ${}_0F_1$, Bailey's cubic transformations et-cetera in terms of Meijer's G-functions, using Mellin-Barnes type contour integral technique with the help of their corresponding hypergeometric form. The results presented here are presumably new.

Integral transforms play an important role in various fields of applied mathematics and physics. The method of solution of problems arising in physics lie at the heart of the use of integral transform. Let $f(t)$ be a real or complex valued function of real variable t, defined on interval $a \leq t \leq b$, which belongs to a certain specified class of functions and let $F(p, t)$ be a definite function of p and t, where p is a complex quantity, whose domain is prescribed, then the integral equation

$$\phi[f(t) : p] = \int_a^b F(p, t) f(t) dt \tag{8.1.1}$$

represents an integral transform $\phi[f(t) : p]$ of the function $f(t)$ with respect to the function $F(p, t)$. Where the class of functions to which $f(t)$ belongs and the domain of p are so prescribed that the integral on the right exists. $F(p, t)$ is called the kernel of the transform $\phi[f(t) : p]$.

If there exists a number "M" independent of t so that $|\frac{f(t)}{g(t)}| \leq M$ as $t \longrightarrow t_0$ in the region R of the complex z-plane, where $g(t) \neq 0$, then we say that $f(t) = O[g(t)]$ as $t \longrightarrow t_0$ in R.

If $\lim_{t \longrightarrow t_0} \frac{f(t)}{g(t)} = 0$, then we say that $f(t) = o[g(t)]$ as $t \longrightarrow t_0$ in R.

Let the function $f(t)$ be piecewise continuous on the closed interval $0 \leq t \leq T$ for every finite $T > 0$. Also let

$$f(t) = O[e^{\alpha t}], \qquad t \to \infty \tag{8.1.2}$$

for some α. Operational images (or operational representations) of many classes of special functions in the classical Laplace transform

$$\mathfrak{L}\{f(t) : p\} = \int_0^\infty e^{-pt} f(t) dt = F(p), \qquad \Re(p) > \alpha, \tag{8.1.3}$$

can be obtained by appealing to Euler's integral

$$\int_0^\infty e^{-pt} t^{\lambda-1} dt = \frac{\Gamma(\lambda)}{p^\lambda} \tag{8.1.4}$$

where $\min\{\Re(\lambda), \Re(p)\} > 0$ or $(\Re(p) = 0, 0 < \Re(\lambda) < 1)$.

The integral (8.1.3) appeared for the first time in Euler's investigation in the year 1737. The regular use of the transformation of the form (8.1.3) began after the publication of P. S. Laplace's book in the year 1812. At the present time the Laplace transformation (8.1.3) is the most usable integral transformation. A complete account or elements of the theory of Laplace transformation can be found in numerous books on Laplace transformation, on operational calculus or on integral transformations. Among them we mention the monographs [60, 84, 93, 210, 211, 235, 236, 237, 243, 245, 254, 295, 335, 336, 338].

For hypergeometric forms of elementary, composite functions, Special functions of Mathematical Physics, Multiple hypergeometric functions and product theorems of hypergeometric functions, we refer the section 1.19, 1.20 and 1.21 of PhD thesis.

8.2 Theorem on Laplace Transforms

Statement:

Let $A \leq B+1$, $\Re(p) > 0$, $\Re(c) > 0$ and k be a positive integer. The Laplace transform of the product of power function with ${}_AF_B[.]$ is given by

$$\mathfrak{L}\left\{t^{c-1}{}_AF_B\left[\begin{array}{c}(a_A);\\(b_B);\end{array} yt^k\right]:p\right\}$$

$$= \frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}k^{\left(c-\frac{1}{2}\right)}G^{A+k,1}_{1+B,A+k}\left(\frac{-p^k}{yk^k}\left|\begin{array}{c}1, b_1,\ldots,b_B\\ a_1,\ldots,a_A,\frac{c}{k},\ldots,\frac{c+k-1}{k}\end{array}\right.\right) \quad (8.2.1)$$

$$= \frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}k^{\left(c-\frac{1}{2}\right)}\times$$

$$\times G^{1,A+k}_{A+k,1+B}\left(\frac{-yk^k}{p^k}\left|\begin{array}{c}1-a_1,\ldots,1-a_A,1+\left(\frac{-c}{k}\right),\ldots,1+\left(\frac{-c-k+1}{k}\right)\\ 0,1-b_1,\ldots,1-b_B\end{array}\right.\right) \quad (8.2.2)$$

provided that the right hand sides of (8.2.1) and (8.2.2) are convergent [See 91, p.207 (2,3,4); 184, p.144 (2,3,4); 235, p.617 (1,2,3,4)], (a_A) abbreviates the array of A parameters given by $a_1, a_2, \ldots, a_A$ with similar interpretation for (b_B) and $a_1, a_2, \ldots, a_A, b_1, b_2, \ldots, b_B \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof: Consider the left hand side of equation (8.2.1):

$$\mathfrak{L}\left\{t^{c-1}{}_AF_B\left[\begin{array}{c}(a_A);\\(b_B);\end{array} yt^k\right]:p\right\} = \int_0^\infty e^{-pt}t^{c-1}{}_AF_B\left[\begin{array}{c}(a_A);\\(b_B);\end{array} yt^k\right]dt$$

$$= \int_0^\infty e^{-pt}t^{c-1}\left(\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\int_L\frac{(-yt^k)^s\Gamma(-s)\prod_{i=1}^{A}\Gamma(a_i+s)}{\prod_{j=1}^{B}\Gamma(b_j+s)}ds\right)dt$$

where L is a suitable Mellin-Barnes type contour.

Changing the order of integration, we get

$$\mathfrak{L}\left\{t^{c-1}{}_AF_B\left[\begin{array}{cc}(a_A); & \\ & yt^k \\ (b_B); & \end{array}\right]:p\right\}$$

$$=\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\int_L\left(\frac{(-y)^s\Gamma(-s)\prod_{i=1}^{A}\Gamma(a_i+s)}{\prod_{j=1}^{B}\Gamma(b_j+s)}\left(\int_0^\infty e^{-pt}t^{c+ks-1}dt\right)\right)ds$$

$$=\frac{1}{p^c}\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\int_L\frac{\left(\frac{-y}{p^k}\right)^s\Gamma(-s)\Gamma(c+ks)\prod_{i=1}^{A}\Gamma(a_i+s)}{\prod_{j=1}^{B}\Gamma(b_j+s)}ds$$

Now using Gauss's multiplication formula (1.2.16) for $\Gamma(c+ks)$, we get

$$\mathfrak{L}\left\{t^{c-1}{}_AF_B\left[\begin{array}{cc}(a_A); & \\ & yt^k \\ (b_B); & \end{array}\right]:p\right\}=\frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}k^{\left(c-\frac{1}{2}\right)}\times$$

$$\times\frac{1}{2\pi\omega}\int_L\frac{\Gamma(-s)\prod_{i=1}^{A}\Gamma(a_i+s)\prod_{q=1}^{k}\Gamma\left(s+\frac{c+q-1}{k}\right)}{\prod_{j=1}^{B}\Gamma(b_j+s)}\left(\frac{-yk^k}{p^k}\right)^s ds$$

$$L\left[t^{c-1}{}_AF_B\left[\begin{array}{cc}(a_A); & \\ & yt^k \\ (b_B); & \end{array}\right];p\right]=\frac{(2\pi)^{\frac{1-k}{2}}}{p^c}\frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{i=1}^{A}\Gamma(a_i)}\frac{k^{\left(c-\frac{1}{2}\right)}}{2\pi w}\times$$

$$\times\int_L\frac{\Gamma(0-s)\prod_{i=1}^{A}\Gamma(1-(1-a_i)+s)\prod_{q=1}^{k}\Gamma\left(1-\left(1+\frac{1-c-q}{k}\right)+s\right)}{\prod_{j=1}^{B}\Gamma(1-(1-b_j)+s)}\left(\frac{-yk^k}{p^k}\right)^s ds \tag{8.2.3}$$

Using definition (1.4.1) and transformation formula (1.4.6) of G-function, we get main results (8.2.2) and (8.2.1) respectively.

8.3 Laplace Transforms of Elementary and Composite Functions

The Laplace transforms of following elementary and composite functions are not found in the existing literature on transform analysis [60, 84, 93, 210, 211, 236, 243, 245, 254, 295, 335, 336]. Making suitable adjustment of parameters and variables in equation (8.2.1) and using the cancellation property (1.4.4), after simplification we can find the following results, valid under the conditions associated with the result (8.2.1).

Case(1): Put $c = m+1,\ A = 0,\ B = 1,\ b_1 = \frac{3}{2},\ y = -\frac{1}{4},\ k = 2m$ in equation (8.2.1) and using corresponding hypergeometric notation of $\sin(t^m)$, we get

$$\mathfrak{L}\left\{\sin(t^m) : p\right\} = \frac{m^{\frac{2m+1}{2}}}{\pi^{m-1} p^{m+1}} G^{2m-1,1}_{1,2m-1}\left(\frac{4p^{2m}}{(2m)^{2m}} \middle| \begin{matrix} 1 \\ \frac{m+1}{2m}, \frac{m+2}{2m}, \ldots, \frac{3m-1}{2m} \end{matrix}\right) \tag{8.3.1}$$

where m is a positive integer.

Case(2): Put $c = 1,\ A = 0,\ B = 1,\ b_1 = \frac{1}{2},\ y = -\frac{1}{4},\ k = 2m$ in equation (8.2.1) and using corresponding hypergeometric notation of $\cos(t^m)$, we get

$$\mathfrak{L}\left\{\cos(t^m) : p\right\} = \frac{\sqrt{m}}{(2\pi)^{m-1} p} G^{2m-1,1}_{1,2m-1}\left(\frac{4p^{2m}}{(2m)^{2m}} \middle| \begin{matrix} 1 \\ \frac{1}{2m}, \frac{2}{2m}, \ldots, \frac{m-1}{2m}, \frac{m+1}{2m}, \ldots, \frac{2m}{2m} \end{matrix}\right) \tag{8.3.2}$$

where m is a positive integer.

Case(3): Put $c = 1,\ A = 0,\ B = 0,\ y = 1,\ k = m$ in equation (8.2.1) and using corresponding hypergeometric notation of $\exp(t^m)$, we get

$$\mathfrak{L}\left\{\exp(t^m) : p\right\} = \frac{\sqrt{m}}{(2\pi)^{\frac{m-1}{2}} p} G^{m,1}_{1,m}\left(-\frac{p^m}{(m)^m} \middle| \begin{matrix} 1 \\ \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} \end{matrix}\right) \tag{8.3.3}$$

where m is a positive integer.

Case(4): Put $c = 2,\ A = 2,\ B = 1,\ a_1 = \frac{1}{2},\ a_2 = 1,\ b_1 = \frac{3}{2},\ y = -1,\ k = 2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\tan^{-1}(t)$, we get

$$\mathfrak{L}\{\tan^{-1}(t) : p\} = \frac{1}{p^2\sqrt{\pi}} G^{3,1}_{1,3}\left(\frac{p^2}{4} \middle| \begin{matrix} 1 \\ \frac{1}{2}, 1, 1 \end{matrix}\right) \tag{8.3.4}$$

Case(5): Put $c=2,\ A=2,\ B=1,\ a_1=\dfrac{1}{2},\ a_2=1,\ b_1=\dfrac{3}{2},\ y=1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\ell n\left(\frac{1+t}{1-t}\right)$, we get

$$\mathfrak{L}\left\{\ell n\left(\frac{1+t}{1-t}\right):p\right\}=\frac{2}{p^2\sqrt{\pi}}G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{1}{2},1,1\end{matrix}\right)\tag{8.3.5}$$

Case(6): Put $c=2,\ A=2,\ B=1,\ a_1=\dfrac{1}{2},\ a_2=\dfrac{1}{2},\ b_1=\dfrac{3}{2},\ y=1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\sin^{-1}(t)$, we get

$$\mathfrak{L}\{\sin^{-1}(t):p\}=\frac{1}{\pi p^2}G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{1}{2},\frac{1}{2},1\end{matrix}\right)\tag{8.3.6}$$

Case(7): Put $c=2,\ A=2,\ B=1,\ a_1=\dfrac{1}{2},\ a_2=\dfrac{1}{2},\ b_1=\dfrac{3}{2},\ y=-1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\sinh^{-1}(t)$, we get

$$\mathfrak{L}\left\{\sinh^{-1}(t):p\right\}=\frac{1}{\pi p^2}G^{3,1}_{1,3}\left(\frac{p^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{1}{2},\frac{1}{2},1\end{matrix}\right)\tag{8.3.7}$$

Case(8): Put $c=3,\ A=3,\ B=2,\ a_1=1,\ a_2=1,\ a_3=1,\ b_1=\dfrac{3}{2},\ b_2=2,\ y=1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\left[\sin^{-1}(t)\right]^2$, we get

$$\mathfrak{L}\left\{\left[\sin^{-1}(t)\right]^2:p\right\}=\frac{2}{p^3}G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\,\begin{matrix}1\\ 1,1,1\end{matrix}\right)\tag{8.3.8}$$

Case(9): Put $c=3,\ A=3,\ B=2,\ a_1=1,\ a_2=1,\ a_3=1,\ b_1=\dfrac{3}{2},\ b_2=2,\ y=-1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\left[\sinh^{-1}(t)\right]^2$, we get

$$\mathfrak{L}\left\{\left[\sinh^{-1}(t)\right]^2:p\right\}=\frac{2}{p^3}G^{3,1}_{1,3}\left(\frac{p^2}{4}\,\middle|\,\begin{matrix}1\\ 1,1,1\end{matrix}\right)\tag{8.3.9}$$

Case(10): Put $c=2,\ A=2,\ B=1,\ a_1=1,\ a_2=1,\ b_1=\dfrac{3}{2},\ y=1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\frac{\sin^{-1}(t)}{\sqrt{(1-t^2)}}$, we get

$$\mathfrak{L}\left\{\frac{\sin^{-1}(t)}{\sqrt{(1-t^2)}}:p\right\}=\frac{1}{p^2}G_{1,3}^{3,1}\left(-\frac{p^2}{4}\middle|\begin{matrix}1\\1,1,1\end{matrix}\right) \tag{8.3.10}$$

Case(11): Put $c=2,\ A=2,\ B=1,\ a_1=1,\ a_2=1,\ b_1=\dfrac{3}{2},\ y=-1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\frac{\sinh^{-1}(t)}{\sqrt{(1+t^2)}}$, we get

$$\mathfrak{L}\left\{\frac{\sinh^{-1}(t)}{\sqrt{(1+t^2)}}:p\right\}=\frac{1}{p^2}G_{1,3}^{3,1}\left(\frac{p^2}{4}\middle|\begin{matrix}1\\1,1,1\end{matrix}\right) \tag{8.3.11}$$

Case(12): Put $c=2,\ A=2,\ B=1,\ a_1=1,\ a_2=1,\ b_1=2,\ y=-1,\ k=1$ in equation (8.2.1) and using corresponding hypergeometric notation of $\ell n(1+t)$, we get

$$\mathfrak{L}\{\ell n(1+t):p\}=\frac{1}{p^2}G_{1,2}^{2,1}\left(p\middle|\begin{matrix}1\\1,1\end{matrix}\right) \tag{8.3.12}$$

Case(13): Put $c=2,\ A=3,\ B=2,\ a_1=1,\ a_2=1,\ a_3=\dfrac{3}{2},\ b_1=2,\ b_2=2,\ y=1,\ k=1$ in equation (8.2.1) and using corresponding hypergeometric notation of $\ell n\left(\frac{2}{1+\sqrt{(1-t)}}\right)$, we get

$$\mathfrak{L}\left\{\ell n\left(\frac{2}{1+\sqrt{(1-t)}}\right):p\right\}=\frac{1}{2p^2\sqrt{\pi}}G_{2,3}^{3,1}\left(-p\middle|\begin{matrix}1,2\\1,1,\frac{3}{2}\end{matrix}\right) \tag{8.3.13}$$

Case(14): Put $c=1,\ A=2,\ B=1,\ a_1=a,\ a_2=a+\dfrac{1}{2},\ b_1=2a+1,\ y=1,\ k=1$ in equation (8.2.1) and using corresponding hypergeometric notation of $\left(\frac{2}{1+\sqrt{(1-t)}}\right)^{2a}$, we get

$$\mathfrak{L}\left\{\left(\frac{2}{1+\sqrt{(1-t)}}\right)^{2a}:p\right\}=\frac{2^{2a}\ a}{p\ \sqrt{\pi}}G_{2,3}^{3,1}\left(-p\middle|\begin{matrix}1,2a+1\\a,a+\frac{1}{2},1\end{matrix}\right) \tag{8.3.14}$$

Case(15): Put $c=1,\ A=2,\ B=1,\ a_1=a,\ a_2=a+\dfrac{1}{2},\ b_1=2a,\ y=1,\ k=1$ in equation (8.2.1) and using corresponding hypergeometric notation of

$\frac{1}{\sqrt{(1-t)}}\left(\frac{2}{1+\sqrt{(1-t)}}\right)^{2a-1}$, we get

$$\mathfrak{L}\left\{\frac{1}{\sqrt{(1-t)}}\left(\frac{2}{1+\sqrt{(1-t)}}\right)^{2a-1}:p\right\}=\frac{2^{(2a-1)}}{p\,\sqrt{\pi}}G^{3,1}_{2,3}\left(-p\left|\begin{array}{c}1,2a\\ a,a+\frac{1}{2},1\end{array}\right.\right) \tag{8.3.15}$$

Case(16): Put $c=1,\ A=2,\ B=1,\ a_1=2a,\ a_2=a+1,\ b_1=a,\ y=1,\ k=1$ in equation (8.2.1) and using corresponding hypergeometric notation of $\frac{1+t}{(1-t)^{2a+1}}$, we get

$$\mathfrak{L}\left\{\frac{1+t}{(1-t)^{2a+1}}:p\right\}=\frac{1}{p\ a\ \Gamma(2a)}G^{3,1}_{2,3}\left(-p\left|\begin{array}{c}1,a\\ 2a,a+1,1\end{array}\right.\right) \tag{8.3.16}$$

where $2a\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(17): Put $c=1,\ A=2,\ B=1,\ a_1=a,\ a_2=b+1,\ b_1=b,\ y=1,\ k=1$ in equation (8.2.1) and using corresponding hypergeometric notation of $\left[1-\left(1-\frac{a}{b}\right)t\right](1-t)^{(-a-1)}$, we get

$$\mathfrak{L}\left\{\left[1-\left(1-\frac{a}{b}\right)t\right](1-t)^{(-a-1)}:p\right\}=\frac{1}{p\ b\ \Gamma(a)}G^{3,1}_{2,3}\left(-p\left|\begin{array}{c}1,b\\ a,b+1,1\end{array}\right.\right) \tag{8.3.17}$$

where $a\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $b\neq 0$.

Case(18): Put $c=1,\ A=2,\ B=1,\ a_1=-a,\ a_2=\dfrac{1}{2}-a,\ b_1=\dfrac{1}{2},\ y=1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $[(1+t)^{2a}+(1-t)^{2a}]$, we get

$$\mathfrak{L}\left\{\left[(1+t)^{2a}+(1-t)^{2a}\right]:p\right\}=\frac{2}{p\ \Gamma(-a)\Gamma\left(\frac{1}{2}-a\right)}G^{3,1}_{1,3}\left(-\frac{p^2}{4}\left|\begin{array}{c}1\\ -a,\frac{1}{2}-a,1\end{array}\right.\right) \tag{8.3.18}$$

where $-a,\frac{1}{2}-a\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(19): Put $c=2,\ A=2,\ B=1,\ a_1=\dfrac{1}{2}-a,\ a_2=1-a,\ b_1=\dfrac{3}{2},\ y=1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $[(1+t)^{2a}-(1-t)^{2a}]$, we get

$$\mathfrak{L}\left\{\left[(1+t)^{2a}-(1-t)^{2a}\right]:p\right\}=\frac{4a}{p^2\ \Gamma(1-a)\Gamma\left(\frac{1}{2}-a\right)}G^{3,1}_{1,3}\left(-\frac{p^2}{4}\left|\begin{array}{c}1\\ \frac{1}{2}-a,1-a,1\end{array}\right.\right) \tag{8.3.19}$$

where $1-a, \frac{1}{2}-a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(20): Put $c=1,\ A=2,\ B=1,\ a_1=b,\ a_2=-b,\ b_1=\dfrac{1}{2},\ y=-1,\ k=2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\left[\left(\sqrt{(1+t^2)}+t\right)^{2b}+\left(\sqrt{(1+t^2)}-t\right)^{2b}\right]$, we get

$$\mathfrak{L}\left\{\left[\left(\sqrt{(1+t^2)}+t\right)^{2b}+\left(\sqrt{(1+t^2)}-t\right)^{2b}\right]:p\right\}=\frac{-2b\sin(b\pi)}{p\pi}G^{3,1}_{1,3}\left(\frac{p^2}{4}\middle|\begin{matrix}1\\ b,-b,1\end{matrix}\right) \tag{8.3.20}$$

where $b\neq 0,\pm1,\pm2,\pm3,\ldots$.

Case(21): Put $c=1,\ A=2,\ B=1,\ a_1=b,\ a_2=1-b,\ b_1=\dfrac{1}{2},\ y=-1,\ k=2$ in equation (8.2.1) and using hypergeometric notation of $\frac{1}{\sqrt{(1+t^2)}}\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-1}+\left(\sqrt{(1+t^2)}-t\right)^{2b-1}\right]$, we get

$$\mathfrak{L}\left\{\frac{1}{\sqrt{(1+t^2)}}\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-1}+\left(\sqrt{(1+t^2)}-t\right)^{2b-1}\right]:p\right\}$$
$$=\frac{2\sin(b\pi)}{p\ \pi}G^{3,1}_{1,3}\left(\frac{p^2}{4}\middle|\begin{matrix}1\\ b,1-b,1\end{matrix}\right) \tag{8.3.21}$$

where $b\neq 0,\pm1,\pm2,\pm3,\ldots$.

Case(22): Put $c=2,\ A=2,\ B=1,\ a_1=b,\ a_2=1-b,\ b_1=\dfrac{3}{2},\ y=-1,\ k=2$ in equation (8.2.1) and using hypergeometric notation of $\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-1}-\left(\sqrt{(1+t^2)}-t\right)^{2b-1}\right]$, we get

$$\mathfrak{L}\left\{\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-1}-\left(\sqrt{(1+t^2)}-t\right)^{2b-1}\right]:p\right\}$$
$$=\frac{2(2b-1)\ \sin(b\pi)}{p^2\ \pi}G^{3,1}_{1,3}\left(\frac{p^2}{4}\middle|\begin{matrix}1\\ b,1-b,1\end{matrix}\right) \tag{8.3.22}$$

where $b\neq 0,\pm1,\pm2,\pm3,\ldots$.

Case(23): Put $c = 2,\ A = 2,\ B = 1,\ a_1 = b,\ a_2 = 2 - b,\ b_1 = \frac{3}{2},\ y = -1,\ k = 2$ in equation (8.2.1) and using hypergeometric notation of $\frac{1}{\sqrt{(1+t^2)}}\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-2} - \left(\sqrt{(1+t^2)}-t\right)^{2b-2}\right]$, we get

$$\mathfrak{L}\left\{\frac{1}{\sqrt{(1+t^2)}}\left[\left(\sqrt{(1+t^2)}+t\right)^{2b-2} - \left(\sqrt{(1+t^2)}-t\right)^{2b-2}\right] : p\right\}$$

$$= \frac{-4\sin(b\ \pi)}{\pi\ p^2} G^{3,1}_{1,3}\left(\frac{p^2}{4}\,\middle|\,\begin{matrix} 1 \\ b, 2-b, 1 \end{matrix}\right) \tag{8.3.23}$$

where $b \neq 0, \pm1, \pm2, \pm3, \ldots$.

Case(24): Put $c = 2,\ A = 2,\ B = 1,\ a_1 = \frac{1-a}{2},\ a_2 = \frac{1+a}{2},\ b_1 = \frac{3}{2},\ y = 1,\ k = 2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\sin\left[a(\sin^{-1}(t))\right]$, we get

$$\mathfrak{L}\left\{\sin\left[a(\sin^{-1}(t))\right] : p\right\} = \frac{a\cos\left(\frac{a\pi}{2}\right)}{p^2\pi} G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\,\begin{matrix} 1 \\ \frac{1-a}{2}, \frac{1+a}{2}, 1 \end{matrix}\right) \tag{8.3.24}$$

where $a \neq \pm1, \pm3, \pm5, \ldots$.

Case(25): Put $c = 1,\ A = 2,\ B = 1,\ a_1 = -\frac{a}{2},\ a_2 = \frac{a}{2},\ b_1 = \frac{1}{2},\ y = 1,\ k = 2$ in equation (8.2.1) and using corresponding hypergeometric notation of $\cos\left[a(\sin^{-1}(t))\right]$, we get

$$\mathfrak{L}\left\{\cos\left[a(\sin^{-1}(t))\right] : p\right\} = \frac{-a\sin\left(\frac{a\pi}{2}\right)}{2\ p\ \pi} G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\,\begin{matrix} 1 \\ -\frac{a}{2}, \frac{a}{2}, 1 \end{matrix}\right) \tag{8.3.25}$$

where $a \neq 0, \pm2, \pm4, \ldots$.

8.4 Laplace Transforms of Special Functions of Mathematical Physics

For convenience, we shall use the notation $\Delta(N;\lambda)$ for array of N parameters given by $\frac{\lambda}{N}, \frac{\lambda+1}{N}, \frac{\lambda+2}{N}, \ldots, \frac{\lambda+N-1}{N}$. The following formulas hold for those suitable values of parameters for which gamma factors of the numerator and denominator are finite.

The Laplace transforms of following Special functions are not found in the available literature on Laplace transforms [60, 84, 93, 210, 211, 236, 237, 243, 245, 254, 295, 335, 338]. Making suitable adjustment of parameters and variables in equation (8.2.1), using the cancellation property (1.4.4), applying reduction formula (1.4.8), Legendre's duplication formula and triplication formula for the product of Gamma functions, after simplification we can find the following results, valid under the conditions associated with the result (8.2.1).

Case(1): Put $c=\mu+\nu+1,\ A=2,\ B=3,\ a_1=\dfrac{\mu+\nu+1}{2},\ a_2=\dfrac{\mu+\nu+2}{2},\ b_1=\mu+1,\ b_2=\nu+1,\ b_3=\mu+\nu+1,\ y=-1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{J_\mu(t)J_\nu(t):p\right\}=\frac{2^{\mu+\nu}}{\pi p^{\mu+\nu+1}}G^{4,1}_{4,4}\left(\frac{p^2}{4}\,\middle|\,\begin{array}{c}1,\mu+1,\nu+1,\mu+\nu+1\\ \frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2},\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2}\end{array}\right)$$

$$=\frac{\Gamma(\mu+\nu+1)}{2^{\mu+\nu}\ p^{\mu+\nu+1}\Gamma(\mu+1)\Gamma(\nu+1)}\,{}_4F_3\left[\begin{array}{c}\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2},\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2};\\ \mu+1,\quad \nu+1,\quad \mu+\nu+1;\end{array}\ -\frac{4}{p^2}\right] \tag{8.4.1}$$

where the product $J_\mu(t)J_\nu(t)$ is given in the monograph [184, p.216 (39)] and $\mu+\nu+1,\mu+1,\nu+1\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(2): Put $c=2\nu+2,\ A=1,\ B=2,\ a_1=\nu+\dfrac{3}{2},\ b_1=\nu+2,\ b_2=2\nu+2,\ y=-1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{J_\nu(t)J_{\nu+1}(t):p\right\}=\frac{2^{2\nu+1}}{\pi p^{2\nu+2}}G^{3,1}_{3,3}\left(\frac{p^2}{4}\,\middle|\,\begin{array}{c}1,\nu+2,2\nu+2\\ \nu+\frac{3}{2},\nu+1,\nu+\frac{3}{2}\end{array}\right)$$

$$=\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\ p^{2\nu+2}\ \Gamma(\nu+2)}\,{}_3F_2\left[\begin{array}{c}\nu+1,\nu+\frac{3}{2},\nu+\frac{3}{2};\\ \nu+2,\quad 2\nu+2\quad;\end{array}\ -\frac{4}{p^2}\right] \tag{8.4.2}$$

where the product $J_\nu(t)J_{\nu+1}(t)$ is given in the monograph [184, p.216 (40)] and $\nu+\frac{3}{2}, \nu+2 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(3): Put $c=2\nu+1,\ A=1,\ B=2,\ a_1=\nu+\dfrac{1}{2},\ b_1=\nu+1,\ b_2=2\nu+1,\ y=-1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{J_\nu^2(t):p\right\}=\frac{2^{2\nu}}{\pi p^{2\nu+1}}G_{2,2}^{2,1}\left(\frac{p^2}{4}\,\middle|\,\begin{matrix}1, 2\nu+1\\ \nu+\frac{1}{2}, \nu+\frac{1}{2}\end{matrix}\right)$$

$$=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\ p^{2\nu+1}\ \Gamma(\nu+1)}\,{}_2F_1\left[\begin{matrix}\nu+\frac{1}{2},\nu+\frac{1}{2};\\ 2\nu+1\quad;\end{matrix}\ -\frac{4}{p^2}\right] \tag{8.4.3}$$

where the product $J_\nu^2(t)$ is given in the monograph [184, p.216 (41)] and $\nu+\frac{1}{2}, \nu+1 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Similarly we can obtain

$$\mathfrak{L}\left\{J_\nu(t)J_{-\nu}(t):p\right\}=\frac{1}{\pi\ p}G_{3,3}^{3,1}\left(\frac{p^2}{4}\,\middle|\,\begin{matrix}1,-\nu+1,\nu+1\\ \frac{1}{2},\quad \frac{1}{2},\quad 1\end{matrix}\right)$$

$$=\frac{\sin(\nu\ \pi)}{\pi\ p\ \nu}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\quad\frac{1}{2},\quad 1\quad;\\ -\nu+1,\nu+1;\end{matrix}\ -\frac{4}{p^2}\right] \tag{8.4.4}$$

where $\nu\neq 0,\pm1,\pm2,\ldots$.

Case(4): Put $c=2\nu+1,\ A=0,\ B=3,\ b_1=\dfrac{\nu+1}{2},\ b_2=\dfrac{\nu+2}{2},\ b_3=\nu+1,\ y=-\dfrac{1}{64},\ k=4$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{J_\nu(t)I_\nu(t):p\right\}=\frac{2^{\nu-\frac{1}{2}}}{\pi p^{2\nu+1}}G_{2,2}^{2,1}\left(\frac{p^4}{4}\,\middle|\,\begin{matrix}1,\nu+1\\ \frac{2\nu+1}{4},\frac{2\nu+3}{4}\end{matrix}\right)$$

$$=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\ p^{2\nu+1}\ \Gamma(\nu+1)}\,{}_2F_1\left[\begin{matrix}\frac{2\nu+1}{4},\frac{2\nu+3}{4};\\ \nu+1\quad;\end{matrix}\ -\frac{4}{p^4}\right] \tag{8.4.5}$$

where the product $J_\nu(t)I_\nu(t)$ is given in the monograph [184, p.216 (43)] and $\nu+\frac{1}{2}, \nu+1 \ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(5): Put $c=m+n+1,\ A=0,\ B=2,\ b_1=m+1,\ b_2=n+1,\ y=-\dfrac{1}{27},\ k=3$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{J_{m,n}(t):p\right\}=\frac{\sqrt{3}}{p^{m+n+1}\ 2\pi}G_{3,3}^{3,1}\left(p^3\,\middle|\,\begin{matrix}1,m+1,n+1\\ \frac{m+n+1}{3},\frac{m+n+2}{3},\frac{m+n+3}{3}\end{matrix}\right)$$

$$= \frac{\Gamma(m+n+1)}{p^{m+n+1}\ 3^{(m+n)}\ \Gamma(m+1)\Gamma(n+1)} {}_3F_2 \left[\begin{array}{l} \Delta(3; m+n+1); \\ m+1, \quad n+1\ ; \end{array} -\frac{1}{p^3} \right] \tag{8.4.6}$$

where $J_{m,n}(t)$ is Hyper-Bessel function of Humbert [95, p.250 (19.7.7); 135; 219, p.102] and $m+n+1, m+1, n+1 \ \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

Case(6): Put $c = m+n+1,\ A = 0,\ B = 2,\ b_1 = m+1,\ b_2 = n+1,\ y = \frac{1}{27},\ k = 3,$ in equation (8.2.1), we have

$$\mathfrak{L}\left\{I_{m,n}(t) : p\right\} = \frac{\sqrt{3}}{p^{m+n+1}\ 2\pi} G_{3,3}^{3,1}\left(-p^3 \left| \begin{array}{c} 1, m+1, n+1 \\ \frac{m+n+1}{3}, \frac{m+n+2}{3}, \frac{m+n+3}{3} \end{array} \right. \right)$$

$$= \frac{\Gamma(m+n+1)}{p^{m+n+1}\ 3^{(m+n)}\ \Gamma(m+1)\Gamma(n+1)} {}_3F_2 \left[\begin{array}{l} \Delta(3; m+n+1); \\ m+1, \quad n+1\ ; \end{array} \frac{1}{p^3} \right] \tag{8.4.7}$$

where $I_{m,n}(t)$ is Modified Hyper-Bessel function of Delerue [76] and $m+n+1, m+1, n+1 \ \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

Case(7): Put $c = 2,\ A = 1,\ B = 1,\ a_1 = \frac{1}{2},\ b_1 = \frac{3}{2},\ y = -1,\ k = 2,$ in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathrm{erf}(t) : p\right\} = \frac{2}{\pi\ p^2} G_{1,2}^{2,1}\left(\frac{p^2}{4} \left| \begin{array}{c} 1 \\ \frac{1}{2}, 1 \end{array} \right. \right) \tag{8.4.8}$$

where $\mathrm{erf}(t)$ is Error function [252, p.36 (6), p.127 (Q.1)].

Case(8): Put $c = \alpha+1,\ A = 2,\ B = 1,\ a_1 = \alpha,\ a_2 = 1-\beta,\ b_1 = 1+\alpha,\ y = 1,\ k = 1,$ in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{B}_t\left(\alpha, \beta\right) : p\right\} = \frac{1}{p^{\alpha+1}\Gamma(1-\beta)} G_{1,2}^{2,1}\left(-p \left| \begin{array}{c} 1 \\ \alpha, 1-\beta \end{array} \right. \right) \tag{8.4.9}$$

where $\mathbf{B}_t\left(\alpha, \beta\right)$ is Incomplete Beta function[326, p.35 (31)] and $1-\beta \ \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

Case(9): Put $c = a+1,\ A = 1,\ B = 1,\ a_1 = a,\ b_1 = 1+a,\ y = -1,\ k = 1,$ in equation (8.2.1), we have

$$\mathfrak{L}\left\{\gamma(a, t) : p\right\} = \frac{1}{p^{a+1}} G_{1,1}^{1,1}\left(p \left| \begin{array}{c} 1 \\ a \end{array} \right. \right)$$

$$= \frac{\Gamma(a)}{p^{a+1}}\ {}_1F_0 \left[\begin{array}{l} a\ ; \\ -\ ; \end{array} -\frac{1}{p} \right] = \frac{\Gamma(a)}{p}(1+p)^{-a} \tag{8.4.10}$$

where $\gamma(a,t)$ is Incomplete Gamma function[184, p.220 (6.2.11.1); 252, p.127 Q.2] and $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(10): Put $c = 1,\ A = 2,\ B = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{2},\ b_1 = 2,\ y = 1,\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{B}(t) : p\} = \frac{1}{4p\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4}\,\middle|\, \begin{matrix} 1,2 \\ \frac{1}{2},\frac{1}{2},\frac{1}{2},1 \end{matrix}\right) \tag{8.4.11}$$

where $\mathbf{B}(t)$ is complete Elliptic integral [92, p.321 13.8(25)].

Case(11): Put $c = 1,\ A = 2,\ B = 1,\ a_1 = \frac{3}{2},\ a_2 = \frac{3}{2},\ b_1 = 3,\ y = 1,\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{C}(t) : p\} = \frac{1}{2p\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4}\,\middle|\, \begin{matrix} 1,3 \\ \frac{3}{2},\frac{3}{2},\frac{1}{2},1 \end{matrix}\right) \tag{8.4.12}$$

where $\mathbf{C}(t)$ is complete Elliptic integral [92, p.321 13.8(25)].

Case(12): Put $c = 1,\ A = 2,\ B = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{3}{2},\ b_1 = 2,\ y = 1,\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{D}(t) : p\} = \frac{1}{2p\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4}\,\middle|\, \begin{matrix} 1,2 \\ \frac{1}{2},\frac{3}{2},\frac{1}{2},1 \end{matrix}\right) \tag{8.4.13}$$

where $\mathbf{D}(t)$ is complete Elliptic integral [92, p.321 13.8(25)].

Case(13): Put $c = 1,\ A = 2,\ B = 1,\ a_1 = \frac{1}{2},\ a_2 = -\frac{1}{2},\ b_1 = 1,\ y = 1,\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{E}(t) : p\} = -\frac{1}{4p\sqrt{\pi}} G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\, \begin{matrix} 1 \\ \frac{1}{2},-\frac{1}{2},\frac{1}{2} \end{matrix}\right) \tag{8.4.14}$$

where $\mathbf{E}(t)$ is complete Elliptic integral of second kind [92, pp.317-318 13.8(2)(6)].

Case(14): Put $c = 1,\ A = 2,\ B = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{2},\ b_1 = 1,\ y = 1,\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{K}(t) : p\} = \frac{1}{2p\sqrt{\pi}} G^{3,1}_{1,3}\left(-\frac{p^2}{4}\,\middle|\, \begin{matrix} 1 \\ \frac{1}{2},\frac{1}{2},\frac{1}{2} \end{matrix}\right) \tag{8.4.15}$$

where $\mathbf{K}(t)$ is complete Elliptic integral of first kind [92, pp.317-318 13.8(1)(5)].

Case(15): Put $c = a,\ A = 2,\ B = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{2},\ b_1 = 1,\ y = b,\ k = 1$, in equation (8.2.1), we have

$$\mathfrak{L}\{t^{a-1}\mathbf{K}(\sqrt{bt}) : p\} = \frac{1}{2\ p^a} G^{3,1}_{2,3}\left(-\frac{p}{b}\middle|\begin{matrix} 1, 1 \\ \frac{1}{2}, \frac{1}{2}, a \end{matrix}\right) \tag{8.4.16}$$

Case(16): Put $c = 4,\ A = 1,\ B = 2,\ a_1 = \frac{3}{4},\ b_1 = \frac{3}{2},\ b_2 = \frac{7}{4},\ y = -\frac{\pi^2}{16},\ k = 4$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{S}(t) : p\} = \frac{2\sqrt{2}}{p^4} G^{3,1}_{1,3}\left(\frac{p^4}{16\ \pi^2}\middle|\begin{matrix} 1 \\ \frac{3}{4}, 1, \frac{5}{4} \end{matrix}\right) \tag{8.4.17}$$

where $\mathbf{S}(t)$ is Fresnel integral [1, p.300 (7.3.2)].

Case(17): Put $c = 4,\ A = 1,\ B = 2,\ a_1 = \frac{3}{4},\ b_1 = \frac{3}{2},\ b_2 = \frac{7}{4},\ y = -\frac{1}{4},\ k = 4$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{S}_1(t) : p\} = \frac{8}{\pi^{\frac{3}{2}} p^4} G^{3,1}_{1,3}\left(\frac{p^4}{64}\middle|\begin{matrix} 1 \\ \frac{3}{4}, 1, \frac{5}{4} \end{matrix}\right) \tag{8.4.18}$$

where $\mathbf{S}_1(t)$ is Fresnel integral [1, p.300 (7.3.4)].

Case(18): Put $c = \frac{5}{2},\ A = 1,\ B = 2,\ a_1 = \frac{3}{4},\ b_1 = \frac{3}{2},\ b_2 = \frac{7}{4},\ y = -\frac{1}{4},\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbf{S}_2(t) : p\} = \frac{1}{2\ \sqrt{\pi}\ p^{\frac{5}{2}}} G^{2,1}_{2,2}\left(p^2\middle|\begin{matrix} 1, \frac{3}{2} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\right)$$

$$= \frac{1}{2\ \sqrt{2}\ p^{\frac{5}{2}}}\ {}_2F_1\left[\begin{matrix} \frac{3}{4}, \frac{5}{4}; \\ \frac{3}{2} \quad ; \end{matrix}\ -\frac{1}{p^2}\right] \tag{8.4.19}$$

where $\mathbf{S}_2(t)$ is Fresnel integral [1, p.300 (7.3.4)].

Case(19): Put $c = 2,\ A = 1,\ B = 2,\ a_1 = \frac{1}{4},\ b_1 = \frac{1}{2},\ b_2 = \frac{5}{4},\ y = -\frac{\pi^2}{16},\ k = 4$, in equation (8.2.1), we have

$$\mathfrak{L}\{\mathbb{C}^*(t) : p\} = \frac{1}{\pi\ \sqrt{2}\ p^2} G^{3,1}_{1,3}\left(\frac{p^4}{16\ \pi^2}\middle|\begin{matrix} 1 \\ \frac{1}{4}, \frac{3}{4}, 1 \end{matrix}\right) \tag{8.4.20}$$

where $\mathbf{C}^*(t)$ is Fresnel integral [1, p.300 (7.3.1)].

Case(20): Put $c=2,\ A=1,\ B=2,\ a_1=\dfrac{1}{4},\ b_1=\dfrac{1}{2},\ b_2=\dfrac{5}{4},\ y=-\dfrac{1}{4},\ k=4,$ in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{C}_1(t):p\right\}=\frac{1}{\pi^{\frac{3}{2}}\ p^2}G_{1,3}^{3,1}\left(\frac{p^4}{64}\,\middle|\,\begin{matrix}1\\ \frac{1}{4},\frac{3}{4},1\end{matrix}\right)\tag{8.4.21}$$

where $\mathbf{C}_1(t)$ is Fresnel integral [1, p.300 (7.3.3)].

Case(21): Put $c=\dfrac{3}{2},\ A=1,\ B=2,\ a_1=\dfrac{1}{4},\ b_1=\dfrac{1}{2},\ b_2=\dfrac{5}{4},\ y=-\dfrac{1}{4},\ k=2,$ in equation (8.2.1), we have

$$\begin{aligned}\mathfrak{L}\left\{\mathbf{C}_2(t):p\right\}&=\frac{1}{2\ \pi^{\frac{1}{2}}\ p^{\frac{3}{2}}}G_{2,2}^{2,1}\left(p^2\,\middle|\,\begin{matrix}1,\frac{1}{2}\\ \frac{1}{4},\frac{3}{4}\end{matrix}\right)\\ &=\frac{1}{\sqrt{2}\ p^{\frac{3}{2}}}\ {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{3}{4};\\ \frac{1}{2}\ \ ;\end{matrix}\ -\frac{1}{p^2}\right]\end{aligned}\tag{8.4.22}$$

where $\mathbf{C}_2(t)$ is Fresnel integral [1, p.300 (7.3.3)].

Case(22): Put $c=2,\ A=1,\ B=2,\ a_1=\dfrac{1}{2},\ b_1=\dfrac{3}{2},\ b_2=\dfrac{3}{2},\ y=-\dfrac{1}{4},\ k=2,$ in equation (8.2.1), we have

$$\begin{aligned}\mathfrak{L}\left\{S_i(t):p\right\}&=\frac{1}{2\ p^2}G_{2,2}^{2,1}\left(p^2\,\middle|\,\begin{matrix}1,\frac{3}{2}\\ \frac{1}{2},1\end{matrix}\right)\\ &=\frac{1}{p^2}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},1;\\ \frac{3}{2}\ \ ;\end{matrix}\ -\frac{1}{p^2}\right]\end{aligned}\tag{8.4.23}$$

where $S_i(t)$ is Sine integral [119, p.886 (8.230); 188, p.347].

Case(23): Put $c=2,\ A=1,\ B=2,\ a_1=\dfrac{1}{2},\ b_1=\dfrac{3}{2},\ b_2=\dfrac{3}{2},\ y=\dfrac{1}{4},\ k=2,$ in equation (8.2.1), we have

$$\begin{aligned}\mathfrak{L}\left\{Shi(t):p\right\}&=\frac{1}{2\ p^2}G_{2,2}^{2,1}\left(-p^2\,\middle|\,\begin{matrix}1,\frac{3}{2}\\ \frac{1}{2},1\end{matrix}\right)\\ &=\frac{1}{p^2}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},1;\\ \frac{3}{2}\ \ ;\end{matrix}\ \frac{1}{p^2}\right]\end{aligned}\tag{8.4.24}$$

where $Shi(t)$ is Hyperbolic Sine integral [119, p.886 (8.221), 188, p.347].

Case(24): Put $c = 2,\ A = q+1,\ B = q,\ a_1 = a_2 = \ldots = a_{q+1} = 1,\ b_1 = b_2 = \cdots = b_q = 2,\ y = 1,\ k = 1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{Li}_q(t) : p\right\} = \frac{1}{p^2} G^{q+1,1}_{q,q+1}\left(-p \,\middle|\, \begin{matrix} 1, \overbrace{2,2,\ldots,2}^{\text{q-1}} \\ \underbrace{1,1,\ldots\ldots,1}_{\text{q+1}} \end{matrix}\right) \tag{8.4.25}$$

Here $\overbrace{2,2,\ldots,2}^{\text{q-1}}$ denotes the numerator parameter "2" is written "$q-1$" times and $\underbrace{1,1,\ldots\ldots,1}_{\text{q+1}}$ denotes the denominator parameter "1" is written "$q+1$" times and $\mathbf{Li}_q(t)$ is Polylogarithm function defined by $\mathbf{Li}_q(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^q}$, (where $|t| < 1$ and $q = 2, 3, 4, \ldots$; when $q = 2$ it is called Dilogarithm function);(See also 91, p.30 (1.11.14)).

Case(25): Put $c = \nu+2,\ A = 1,\ B = 2,\ a_1 = 1,\ b_1 = \frac{3}{2},\ b_2 = \nu + \frac{3}{2},\ y = -\frac{1}{4},\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{H}_\nu(t) : p\right\} = \frac{1}{p^{\nu+2}\sqrt{\pi}} G^{3,1}_{3,3}\left(p^2 \,\middle|\, \begin{matrix} 1, \frac{3}{2}, \nu+\frac{3}{2} \\ 1, \frac{\nu+2}{2}, \frac{\nu+3}{2} \end{matrix}\right)$$

$$= \frac{\Gamma(\nu+2)}{p^{\nu+2}\ \sqrt{\pi}\ 2^\nu\ \Gamma\left(\nu+\frac{3}{2}\right)}\ {}_3F_2\left[\begin{matrix} 1, \Delta(2; \nu+2); \\ \frac{3}{2}, \quad \nu+\frac{3}{2}\ ; \end{matrix}\ -\frac{1}{p^2}\right] \tag{8.4.26}$$

where $\mathbf{H}_\nu(t)$ is Struve function [1, p.496 (12.1.3); 184, p.217 (6.2.9.3)] and $\nu + 2, \nu + \frac{3}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(26): Put $c = \nu+2,\ A = 1,\ B = 2,\ a_1 = 1,\ b_1 = \frac{3}{2},\ b_2 = \nu + \frac{3}{2},\ y = \frac{1}{4},\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{L}_\nu(t) : p\right\} = \frac{1}{p^{\nu+2}\sqrt{\pi}} G^{3,1}_{3,3}\left(-p^2 \,\middle|\, \begin{matrix} 1, \frac{3}{2}, \nu+\frac{3}{2} \\ 1, \frac{\nu+2}{2}, \frac{\nu+3}{2} \end{matrix}\right)$$

$$= \frac{\Gamma(\nu+2)}{p^{\nu+2}\ \sqrt{\pi}\ 2^\nu\ \Gamma\left(\nu+\frac{3}{2}\right)}\ {}_3F_2\left[\begin{matrix} 1, \Delta(2; \nu+2); \\ \frac{3}{2}, \quad \nu+\frac{3}{2}\ ; \end{matrix}\ \frac{1}{p^2}\right] \tag{8.4.27}$$

where $\mathbf{L}_\nu(t)$ is Modified Struve function [1, p.498 (12.2.1); 184, p.217 (6.2.9.5)] and $\nu + 2, \nu + \frac{3}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(27): Put $c = \mu+2,\ A = 1,\ B = 2,\ a_1 = 1,\ b_1 = \dfrac{\mu-\nu+3}{2},\ b_2 = \dfrac{\mu+\nu+3}{2},\ y = -\dfrac{1}{4},\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{s}_{\mu,\nu}(t) : p\right\} = \frac{2^{\mu-1}}{p^{\mu+2}\sqrt{\pi}}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)G^{3,1}_{3,3}\left(p^2\left|\begin{matrix}1, \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}\\ 1, \frac{\mu+2}{2}, \frac{\mu+3}{2}\end{matrix}\right.\right)$$

$$= \frac{\Gamma(\mu+2)}{p^{\mu+2}\ [(\mu+1)^2-\nu^2]}\ {}_3F_2\left[\begin{matrix}1, \Delta(2;\mu+2); & \\ \frac{\mu-\nu+3}{2}, \quad \frac{\mu+\nu+3}{2} \quad ; & \end{matrix} -\frac{1}{p^2}\right] \tag{8.4.28}$$

where $\mathbf{s}_{\mu,\nu}(t)$ is Lommel function [184, p.217 (6.2.9.1)] and $\frac{\mu-\nu+1}{2}, \frac{\mu+\nu+1}{2}, \mu+2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(28): Put $c = 1,\ A = 0,\ B = 3,\ b_1 = \dfrac{1}{2},\ b_2 = \dfrac{1}{2},\ b_3 = 1,\ y = -\dfrac{1}{256},\ k = 4$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{ber(t) : p\right\} = \frac{1}{p\ \sqrt{2\pi}}G^{2,1}_{2,2}\left(p^4\left|\begin{matrix}1, \frac{1}{2}\\ \frac{1}{4}, \frac{3}{4}\end{matrix}\right.\right)$$

$$= \frac{1}{p}\ {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{3}{4}; & \\ \frac{1}{2} \quad ; & \end{matrix} -\frac{1}{p^4}\right] \tag{8.4.29}$$

where $ber(t)$ is Kelvin's function [119, p.944 (8.564.1)].

Case(29): Put $c = 3,\ A = 0,\ B = 3,\ b_1 = \dfrac{3}{2},\ b_2 = \dfrac{3}{2},\ b_3 = 1,\ y = -\dfrac{1}{256},\ k = 4$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{bei(t) : p\right\} = \frac{1}{p^3\ \sqrt{2\pi}}G^{2,1}_{2,2}\left(p^4\left|\begin{matrix}1, \frac{3}{2}\\ \frac{3}{4}, \frac{5}{4}\end{matrix}\right.\right)$$

$$= \frac{1}{2\ \pi\ p^3}\ {}_2F_1\left[\begin{matrix}\frac{3}{4}, \frac{5}{4}; & \\ \frac{3}{2} \quad ; & \end{matrix} -\frac{1}{p^4}\right] \tag{8.4.30}$$

where $bei(t)$ is Kelvin's function [119 p.944 (8.564.2)].

Case(30): Put $c = 1,\ A = q+1,\ B = q,\ a_1 = 1,\ a_2 = a_3 = \ldots = a_{q+1} = a,\ b_1 = b_2 = \cdots = b_q = a+1,\ y = 1,\ k = 1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{\Phi}(t,q,a) : p\right\} = \frac{1}{p}G^{q+2,1}_{1+q,q+2}\left(-p\left|\begin{matrix}1, \overbrace{\text{a+1,a+1,}\ldots\text{,a+1}}^{\text{q}}\\ 1, \underbrace{\text{a,a,}\ldots\text{,a}}_{\text{q}}, 1\end{matrix}\right.\right) \tag{8.4.31}$$

Here $\overbrace{\text{a+1,a+1,...,a+1}}^{\text{q}}$ denotes the numerator parameter "$a+1$" is written "q" times and $\underbrace{\text{a,a,...,a}}_{\text{q}}$ denotes the denominator parameter "a" is written "q" times; q is positive integer and $\mathbf{\Phi}(t,q,a)$ is Lerch's transcendent given in the monographs [119, p.1039; 188, p.32 (1.6)].

8.5 Laplace Transforms of Multiple Hypergeometric Functions

The Laplace transforms of following Multiple hypergeometric functions are not found in the available literature on Laplace transforms [60, 84, 93, 210, 211, 236, 237, 254, 295, 335, 338]. Making suitable adjustment of parameters and variables in equation (8.2.1), using the cancellation property (1.4.4), applying reduction formula (1.4.8), Legendre's duplication formula and triplication formula for the product of Gamma functions, after simplification we can find the following results, valid under the conditions associated with the result (8.2.1).

Case(1): Put $c=1,\ A=3,\ B=2,\ a_1=a,\ a_2=b,\ a_3=c,\ b_1=1+a-b,\ b_2=1+a-c,\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{(1-t)^{-a}{}_3F_2\left[\begin{array}{c}\frac{a}{2},\ \frac{a+1}{2}\ ,\ 1+a-b-c;\\ 1+a-b\ ,\quad 1+a-c\ ;\end{array}\ -\frac{4t}{(1-t)^2}\right]:p\right\}$$

$$=\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{p\ \Gamma(a)\Gamma(b)\Gamma(c)}G^{4,1}_{3,4}\left(-p\left|\begin{array}{c}1,1+a-b,1+a-c\\ a,b,c,1\end{array}\right.\right) \tag{8.5.1}$$

where $1+a-b,\ 1+a-c,\ a,\ b,\ c\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$ and ${}_3F_2[.]$ is Clausenian hypergeometric functions.

Case(2): Put $c=1,\ A=3,\ B=2,\ a_1=a,\ a_2=2b-a-1,\ a_3=a+2-2b,\ b_1=b,\ b_2=a-b+\dfrac{3}{2},\ y=\dfrac{1}{4},\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{(1-t)^{-a}{}_3F_2\left[\begin{array}{c}\frac{a}{3},\ \frac{a+1}{3}\ ,\ \frac{a+2}{3};\\ b\ ,\ a-b+\frac{3}{2}\ ;\end{array}\ -\frac{27t}{4(1-t)^3}\right]:p\right\}$$

$$= \frac{\Gamma(b)\Gamma\left(a-b+\frac{3}{2}\right)}{p\ \Gamma(a)\Gamma(2b-a-1)\Gamma(a+2-2b)} G^{4,1}_{3,4}\left(-4p \left| \begin{matrix} 1, b, a-b+\frac{3}{2} \\ a, 2b-a-1, a+2-2b, 1 \end{matrix} \right.\right) \tag{8.5.2}$$

where $a,\ b,\ a-b+\frac{3}{2},\ 2b-a-1, a+2-2b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(3): Put $c=1,\ A=3,\ B=2,\ a_1=a,\ a_2=b-\dfrac{1}{2},\ a_3=1+a-b,\ b_1=2b,\ b_2=2+2a-2b,\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\left(1-\frac{t}{4}\right)^{-a} {}_3F_2\left[\begin{matrix} \frac{a}{3},\ \frac{a+1}{3}\ ,\ \frac{a+2}{3}; \\ b\ ,\ a-b+\frac{3}{2}\ ; \end{matrix}\ \frac{27t^2}{(4-t)^3}\right] : p\right\}$$

$$= \frac{2^{(2a)}\ \left(b-\frac{1}{2}\right)\ \Gamma(b)\Gamma\left(a-b+\frac{3}{2}\right)}{\pi\ p\ \Gamma(a)} G^{4,1}_{3,4}\left(-p \left| \begin{matrix} 1, 2b, 2+2a-2b \\ a, b-\frac{1}{2}, 1+a-b, 1 \end{matrix} \right.\right) \tag{8.5.3}$$

where $a,\ b,\ a-b+\frac{3}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(4): Put $c=1,\ A=2,\ B=1,\ a_1=a,\ a_2=b,\ b_1=2a,\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\left(1-\frac{t}{2}\right)^{-b} {}_2F_1\left[\begin{matrix} \frac{b}{2}, \frac{b+1}{2}; \\ a+\frac{1}{2}\ ; \end{matrix}\ \left(\frac{t}{2-t}\right)^2\right] : p\right\} = \frac{2^{(2a-1)}\ \Gamma\left(a+\frac{1}{2}\right)}{\sqrt{\pi}\ p\ \Gamma(b)} G^{3,1}_{2,3}\left(-p \left| \begin{matrix} 1, 2a \\ a, b, 1 \end{matrix} \right.\right) \tag{8.5.4}$$

where $a+\frac{1}{2},\ b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and ${}_2F_1[.]$ is Gauss ordinary hypergeometric function.

Case(5): Put $c=1,\ A=2,\ B=1,\ a_1=a,\ a_2=a+\dfrac{1}{2}-b,\ b_1=b+\dfrac{1}{2},\ y=1,\ k=2$, in result (8.2.1), we have

$$\mathfrak{L}\left\{(1+t)^{-2a}\ {}_2F_1\left[\begin{matrix} a, b\ ; \\ 2b\ \ ; \end{matrix}\ \frac{4t}{(1+t)^2}\right] : p\right\}$$

$$= \frac{\Gamma\left(b+\frac{1}{2}\right)}{p\Gamma(a)\Gamma\left(a-b+\frac{1}{2}\right)\sqrt{\pi}} G^{4,1}_{2,4}\left(-\frac{p^2}{4} \left| \begin{matrix} 1, b+\frac{1}{2} \\ a, a-b+\frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \right.\right) \tag{8.5.5}$$

where $a-b+\frac{1}{2},\ a,\ b+\frac{1}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(6): Put $c=1,\ A=2,\ B=1,\ a_1=2a,\ a_2=2b,\ b_1=a+b+\dfrac{1}{2},\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_2F_1\left[\begin{matrix} a, \quad b\ ; \\ a+b+\frac{1}{2}\ ; \end{matrix}\ 4t(1-t)\right] : p\right\} = \frac{\Gamma\left(a+b+\frac{1}{2}\right)}{\Gamma(2a)\Gamma(2b)p} G^{3,1}_{2,3}\left(-p \left| \begin{matrix} 1, a+b+\frac{1}{2} \\ 2a, 2b, 1 \end{matrix} \right.\right) \tag{8.5.6}$$

where $a+b+\frac{1}{2},\ 2a,\ 2b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(7): Put $c=1,\ A=3,\ B=2,\ a_1=2a,\ a_2=2b,\ a_3=a+b,\ b_1=2a+2b-1,\ b_2=a+b+\dfrac{1}{2},\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_2F_1\left[\begin{array}{c}a,\quad b\ ;\\ a+b-\frac{1}{2}\ ;\end{array} t\right]{}_2F_1\left[\begin{array}{c}a,\quad b\ ;\\ a+b+\frac{1}{2}\ ;\end{array} t\right]:p\right\}$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}\right)}{p\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)}G^{4,1}_{3,4}\left(-p\,\middle|\,\begin{array}{c}1,2a+2b-1,a+b+\frac{1}{2}\\ 2a,2b,a+b,1\end{array}\right) \tag{8.5.7}$$

where $a+b+\frac{1}{2},\ a+b-\frac{1}{2},\ a,\ b,\ a+\frac{1}{2},\ b+\frac{1}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(8): Put $c=1,\ A=3,\ B=2,\ a_1=2a,\ a_2=2b-1,\ a_3=a+b-1,\ b_1=2a+2b-2,\ b_2=a+b-\dfrac{1}{2},\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_2F_1\left[\begin{array}{c}a,\quad b\ ;\\ a+b-\frac{1}{2}\ ;\end{array} t\right]{}_2F_1\left[\begin{array}{c}a,\quad b-1\ ;\\ a+b-\frac{1}{2}\ ;\end{array} t\right]:p\right\}$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left(a+b-\frac{1}{2}\right)}{p\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b-\frac{1}{2}\right)\Gamma(b)}G^{4,1}_{3,4}\left(-p\,\middle|\,\begin{array}{c}1,2a+2b-2,a+b-\frac{1}{2}\\ 2a,2b-1,a+b-1,1\end{array}\right) \tag{8.5.8}$$

where $a+b-\frac{1}{2},\ a,\ b,\ a+\frac{1}{2},\ b-\frac{1}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(9): Put $c=1,\ A=3,\ B=2,\ a_1=a-b+\dfrac{1}{2},\ a_2=b-a+\dfrac{1}{2},\ a_3=\dfrac{1}{2},\ b_1=a+b+\dfrac{1}{2},\ b_2=\dfrac{3}{2}-a-b,\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_2F_1\left[\begin{array}{c}a,\quad b\ ;\\ a+b+\frac{1}{2}\ ;\end{array} t\right]{}_2F_1\left[\begin{array}{c}\frac{1}{2}-a,\ \frac{1}{2}-b\ ;\\ \frac{3}{2}-a-b\quad ;\end{array} t\right]:p\right\}$$

$$=\frac{\Gamma(\frac{3}{2}-a-b)\Gamma(a+b+\frac{1}{2})}{p\Gamma(b-a+\frac{1}{2})\Gamma(a-b+\frac{1}{2})\sqrt{\pi}}G^{4,1}_{3,4}\left(-p\,\middle|\,\begin{array}{c}1,a+b+\frac{1}{2},\frac{3}{2}-a-b\\ a-b+\frac{1}{2},b-a+\frac{1}{2},\frac{1}{2},1\end{array}\right) \tag{8.5.9}$$

where $a+b+\frac{1}{2},\ \frac{3}{2}-a-b,\ b-a+\frac{1}{2},\ a-b+\frac{1}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(10): Put $c=1,\ A=3,\ B=2,\ a_1=2a,\ a_2=2b,\ a_3=a+b,\ b_1=2a+2b,\ b_2=a+b+\dfrac{1}{2},\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\left({}_2F_1\left[\begin{array}{c}a,\quad b\ ;\\ a+b+\frac{1}{2}\ ;\end{array} t\right]\right)^2:p\right\}$$

$$= \frac{2\sqrt{\pi}\,\Gamma\left(a+b+\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}\right)}{p\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)} G^{4,1}_{3,4}\left(-p\left|\begin{array}{c} 1, 2a+2b, a+b+\frac{1}{2} \\ 2a, 2b, a+b, 1 \end{array}\right.\right) \tag{8.5.10}$$

where $a+b+\frac{1}{2},\ a,\ b,\ a+\frac{1}{2},\ b+\frac{1}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(11): Put $c=1,\ A=2,\ B=1,\ a_1=a,\ a_2=b+c,\ b_1=d,\ y=1,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_1[a;b,c;d;t,t]:p\right\} = \frac{\Gamma(d)}{p\Gamma(a)\Gamma(b+c)} G^{3,1}_{2,3}\left(-p\left|\begin{array}{c} 1, d \\ a, b+c, 1 \end{array}\right.\right) \tag{8.5.11}$$

where $d,\ a,\ b+c\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathbf{F_1}[.]$ is Appell's function of first kind [326, p.53 (1.6.4)].

Case(12): Put $c=1,\ A=3,\ B=2,\ a_1=\frac{a}{2},\ a_2=\frac{a+1}{2},\ a_3=b,\ b_1=\frac{c}{2},\ b_2=\frac{c+1}{2},\ y=1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_1[a;b,b;c;t,-t]:p\right\} = \frac{2^{(a-c)}\ \Gamma(c)}{\sqrt{\pi}\ p\ \Gamma(a)\ \Gamma(b)} G^{5,1}_{3,5}\left(-\frac{p^2}{4}\left|\begin{array}{c} 1, \Delta(2;c) \\ \Delta(2;a), b, \frac{1}{2}, 1 \end{array}\right.\right) \tag{8.5.12}$$

where $c,\ a,\ b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(13): Put $c=1,\ A=4,\ B=3,\ a_1=\frac{a}{2},\ a_2=\frac{a+1}{2},\ a_3=b,\ a_4=c-b,\ b_1=c,\ b_2=\frac{c}{2},\ b_3=\frac{c+1}{2},\ y=1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_2[a;b,b;c,c;t,-t]:p\right\} = \frac{2^{(a-c)}\ [\Gamma(c)]^2}{\sqrt{\pi}\ p\ \Gamma(a)\ \Gamma(b)\ \Gamma(c-b)} G^{6,1}_{4,6}\left(-\frac{p^2}{4}\left|\begin{array}{c} 1, c, \Delta(2;c) \\ \Delta(2;a), b, c-b, \frac{1}{2}, 1 \end{array}\right.\right) \tag{8.5.13}$$

where $c,\ a,\ b,\ c-b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathbf{F_2}[.]$ is Appell's function of second kind [326, p.53 (1.6.5)].

Case(14): Put $c=1,\ A=4,\ B=3,\ a_1=\frac{a}{2},\ a_2=\frac{a+1}{2},\ a_3=\frac{\lambda+\mu}{2},\ a_4=\frac{\lambda+\mu+1}{2},\ b_1=\lambda+\mu,\ b_2=\lambda+\frac{1}{2},\ b_3=\mu+\frac{1}{2},\ y=1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_2[a;\lambda,\mu;2\lambda,2\mu;t,-t]:p\right\} = \frac{2^{(\lambda+\mu+a-2)}\ \Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)}{\pi^{\frac{3}{2}}\ p\ \Gamma(a)} \times$$

$$\times\ G^{6,1}_{4,6}\left(-\frac{p^2}{4}\left|\begin{array}{c} 1, \lambda+\mu, \lambda+\frac{1}{2}, \mu+\frac{1}{2} \\ \Delta(2;a), \Delta(2;\lambda+\mu), \frac{1}{2}, 1 \end{array}\right.\right) \tag{8.5.14}$$

where $a,\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2}\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(15): Put $c=1,\ A=4,\ B=3,\ a_1=a,\ a_2=b,\ a_3=\dfrac{a+b}{2},\ a_4=\dfrac{a+b+1}{2},\ b_1=a+b,\ b_2=\dfrac{c}{2},\ b_3=\dfrac{c+1}{2},\ y=1,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_3[a,a;b,b;c;t,-t]:p\right\}=\frac{2^{(a+b-c)}\Gamma(c)}{\sqrt{\pi}\ p\ \Gamma(a)\Gamma(b)}G_{4,6}^{6,1}\left(-\frac{p^2}{4}\middle|\begin{array}{c}1,a+b,\Delta(2;c)\\a,b,\Delta(2;a+b),\frac{1}{2},1\end{array}\right) \tag{8.5.15}$$

where $c,\ a,\ b\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathbf{F_3}[.]$ is Appell's function of third kind [326, p.53 (1.6.6)].

Case(16): Put $c=1,\ A=4,\ B=3,\ a_1=a,\ a_2=b,\ a_3=\dfrac{c+d}{2},\ a_4=\dfrac{c+d-1}{2},\ b_1=c,\ b_2=d,\ b_3=c+d-1,\ y=4,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_4[a,b;c,d;t,t]:p\right\}=\frac{2^{(c+d-2)}\ \Gamma(c)\Gamma(d)}{\sqrt{\pi}\ p\ \Gamma(a)\Gamma(b)}G_{4,5}^{5,1}\left(-\frac{p}{4}\middle|\begin{array}{c}1,c,d,c+d-1\\a,b,\Delta(2;c+d-1),1\end{array}\right) \tag{8.5.16}$$

where $c,\ d,\ a,\ b\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathbf{F_4}[.]$ is Appell's function of fourth kind [326, p.53 (1.6.7)].

Case(17): Put $c=1,\ A=4,\ B=3,\ a_1=\dfrac{a}{2},\ a_2=\dfrac{a+1}{2},\ a_3=\dfrac{b}{2},\ a_4=\dfrac{b+1}{2},\ b_1=c,\ b_2=\dfrac{c}{2},\ b_3=\dfrac{c+1}{2},\ y=-4,\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\mathbf{F}_4[a,b;c,c;t,-t]:p\right\}=\frac{2^{(a+b-c-1)}\ [\Gamma(c)]^2}{\pi\ p\ \Gamma(a)\ \Gamma(b)}G_{4,6}^{6,1}\left(\frac{p^2}{16}\middle|\begin{array}{c}1,c,\Delta(2;c)\\\Delta(2;a),\Delta(2;b),\frac{1}{2},1\end{array}\right) \tag{8.5.17}$$

where $c,\ a,\ b\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(18): Put $c=1,\ A=1,\ B=1,\ a_1=a,\ b_1=b,\ y=1,\ k=1$, in equation (8.2.1), we have

$$\begin{aligned}\mathfrak{L}\left\{e^t{}_1F_1\left[\begin{array}{ll}b-a\ ; & \\ b\ \ \ \ ; & -t\end{array}\right]:p\right\}&=\frac{\Gamma(b)}{p\Gamma(a)}G_{2,2}^{2,1}\left(-p\middle|\begin{array}{c}1,b\\a,1\end{array}\right)\\&=\frac{1}{p}\ {}_2F_1\left[\begin{array}{ll}a,\ 1; & \dfrac{1}{p}\\ b\ \ \ ; & \end{array}\right]\end{aligned} \tag{8.5.18}$$

where $a,\ b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and ${}_1F_1[.]$ is Kummer's confluent hypergeometric function.

Case(19): Put $c = 1,\ A = 1,\ B = 1,\ a_1 = a,\ b_1 = 2a,\ y = 2,\ k = 1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{e^t{}_0F_1\left[\begin{matrix} - & ; & \frac{t^2}{4} \\ a+\frac{1}{2}; & & \end{matrix}\right] : p\right\} = \frac{2^{(2a-1)}\,\Gamma\left(a+\frac{1}{2}\right)}{p\,\sqrt{\pi}} G^{2,1}_{2,2}\left(-\frac{p}{2}\middle|\begin{matrix}1,2a\\ a,1\end{matrix}\right)$$
$$= \frac{1}{p}\ {}_2F_1\left[\begin{matrix} a,\ 1; & \frac{2}{p} \\ 2a\ ; & \end{matrix}\right] \tag{8.5.19}$$

where $a+\frac{1}{2}\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(20): Put $c = 1,\ A = 2,\ B = 3,\ a_1 = a,\ a_2 = b-a,\ b_1 = b,\ b_2 = \frac{b}{2},\ b_3 = \frac{b+1}{2},\ y = \frac{1}{4},\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_1F_1\left[\begin{matrix} a\ ; & \\ b\ ; & t \end{matrix}\right]{}_1F_1\left[\begin{matrix} a\ ; & \\ b\ ; & -t \end{matrix}\right] : p\right\} = \frac{[\Gamma(b)]^2}{2^{(b-1)}\ p\ \Gamma(a)\Gamma(b-a)} G^{4,1}_{4,4}\left(-p^2\middle|\begin{matrix}1,b,\Delta(2;b)\\ a,b-a,\frac{1}{2},1\end{matrix}\right)$$
$$= \frac{1}{p}\ {}_4F_3\left[\begin{matrix} a,b-a,\frac{1}{2},1; & \frac{1}{p^2} \\ b,\quad \Delta(2;b)\ \ ; & \end{matrix}\right] \tag{8.5.20}$$

where $a,\ b,\ b-a\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(21): Put $c = 1,\ A = 3,\ B = 8,\ a_1 = \frac{a+b-1}{2},\ a_2 = \frac{a+b}{3},\ a_3 = \frac{a+b+1}{3},\ b_1 = a,\ b_2 = b,\ b_3 = \frac{a}{2},\ b_4 = \frac{b}{2},\ b_5 = \frac{a+1}{2},\ b_6 = \frac{b+1}{2},\ b_7 = \frac{a+b-1}{2},\ b_8 = \frac{a+b}{2},\ y = -\frac{27}{64},\ k = 2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_0F_2\left[\begin{matrix} - & ; & \\ a,\ b & ; & t \end{matrix}\right]{}_0F_2\left[\begin{matrix} - & ; & \\ a,\ b & ; & -t \end{matrix}\right] : p\right\} = \frac{3^{\left(a+b-\frac{3}{2}\right)}\ [\Gamma(a)]^2\ [\Gamma(b)]^2}{p\ 2^{(2a+2b-3)}}\times$$
$$\times G^{5,1}_{9,5}\left(\frac{16p^2}{27}\middle|\begin{matrix}1,a,b,\Delta(2;a),\Delta(2;b),\Delta(2;a+b-1)\\ \Delta(3;a+b-1),\frac{1}{2},1\end{matrix}\right)$$
$$= \frac{1}{p}\ {}_5F_8\left[\begin{matrix} \Delta(3;a+b-1), & \frac{1}{2}, & 1 & ; & -\frac{27}{16\ p^2} \\ a,b,\Delta(2;a),\Delta(2;b),\Delta(2;a+b-1); & & & & \end{matrix}\right] \tag{8.5.21}$$

where $a,\ b\ \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(22): Put $c=1,\ A=2,\ B=3,\ a_1=\dfrac{a+b}{2},\ a_2=\dfrac{a+b+1}{2},\ b_1=a+\dfrac{1}{2},\ b_2=b+\dfrac{1}{2},\ b_3=a+b,\ y=\dfrac{1}{4},\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_1F_1\left[\begin{array}{c} a\ ; \\ 2a\ ; \end{array} t\right]{}_1F_1\left[\begin{array}{c} b\ ; \\ 2b\ ; \end{array} -t\right]:p\right\}=\frac{2^{(a+b-1)}\ \Gamma\big(a+\frac{1}{2}\big)\Gamma\big(b+\frac{1}{2}\big)}{\pi\ p}\times$$

$$\times G^{4,1}_{4,4}\left(-p^2\left|\begin{array}{c} 1,a+\frac{1}{2},b+\frac{1}{2},a+b \\ \Delta(2;a+b),\frac{1}{2},1 \end{array}\right.\right)$$

$$=\frac{1}{p}\ {}_4F_3\left[\begin{array}{c} \Delta(2;a+b),\ \frac{1}{2},\ 1; \\ a+\frac{1}{2},b+\frac{1}{2},a+b; \end{array} \frac{1}{p^2}\right] \tag{8.5.22}$$

where $a+\frac{1}{2},\ b+\frac{1}{2}\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(23): Put $c=1,\ A=2,\ B=7,\ a_1=3b-\dfrac{1}{4},\ a_2=3b+\dfrac{1}{4},\ b_1=6b,\ b_2=2b,\ b_3=2b+\dfrac{1}{3},\ b_4=2b+\dfrac{2}{3},\ b_5=4b-\dfrac{1}{3},\ b_6=4b,\ b_7=4b+\dfrac{1}{3},\ y=\dfrac{64}{729},\ k=3$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_0F_1\left[\begin{array}{c} -\ ; \\ 6b\ ; \end{array} t\right]{}_0F_1\left[\begin{array}{c} -\ ; \\ 6b\ ; \end{array} \omega t\right]{}_0F_1\left[\begin{array}{c} -\ ; \\ 6b\ ; \end{array} \omega^2 t\right]:p\right\}=\frac{2^{\left(18b-\frac{5}{2}\right)}[\Gamma(6b)]^3}{3^{\left(18b-\frac{5}{2}\right)}\ p}\times$$

$$\times G^{5,1}_{8,5}\left(-\frac{27p^3}{64}\left|\begin{array}{c} 1,6b,\Delta(3;6b),\Delta(3;12b-1) \\ \Delta(2;6b-\frac{1}{2}),\Delta(3;1) \end{array}\right.\right)$$

$$=\frac{1}{p}\ {}_5F_7\left[\begin{array}{c} \Delta(2;6b-\frac{1}{2}),\qquad \Delta(3;1); \\ 6b,\Delta(3;6b),\Delta(3;12b-1); \end{array} \frac{64}{27\ p^3}\right] \tag{8.5.23}$$

where ω is the cube root of unity ($\omega=e^{\frac{2\pi i}{3}}$) and $6b\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(24): Put $c=1,\ A=0,\ B=3,\ b_1=a,\ b_2=\dfrac{a}{2},\ b_3=\dfrac{a+1}{2},\ y=-\dfrac{1}{4},\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_0F_1\left[\begin{array}{c} -; \\ a\ ; \end{array} t\right]{}_0F_1\left[\begin{array}{c} -; \\ a\ ; \end{array} -t\right]:p\right\}=\frac{[\Gamma(a)]^2}{p\ 2^{(a-1)}}G^{2,1}_{4,2}\left(p^2\left|\begin{array}{c} 1,a,\Delta(2;a) \\ \frac{1}{2},1 \end{array}\right.\right)$$

$$=\frac{1}{p}\ {}_2F_3\left[\begin{array}{c} \frac{1}{2},\qquad 1\ ; \\ a,\Delta(2;a); \end{array} -\frac{1}{p^2}\right] \tag{8.5.24}$$

where $a\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(25): Put $c=1,\ A=2,\ B=3,\ a_1=\dfrac{a+b}{2},\ a_2=\dfrac{a+b-1}{2},\ b_1=a,\ b_2=b,\ b_3=a+b-1,\ y=4,\ k=1$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{{}_0F_1\left[\begin{matrix}-;\\ a\ ;\end{matrix}\ t\right]{}_0F_1\left[\begin{matrix}-;\\ b\ ;\end{matrix}\ t\right]:p\right\}=\frac{2^{(a+b-2)}\ \Gamma(a)\Gamma(b)}{\sqrt{\pi}\ \ p}G^{3,1}_{4,3}\left(-\frac{p}{4}\ \middle|\ \begin{matrix}1,a,b,a+b-1\\ \Delta(2;a+b-1),1\end{matrix}\right)$$

$$=\frac{1}{p}\ {}_3F_3\left[\begin{matrix}\Delta(2;a+b-1),1\ ;\\ a,\quad b,\quad a+b-1\quad ;\end{matrix}\ \frac{4}{p}\right] \tag{8.5.25}$$

where $a,\ b\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Case(26): Put $c=1,\ A=1,\ B=2,\ a_1=a,\ b_1=a+\dfrac{1}{2},\ b_2=2a,\ y=\dfrac{1}{4},\ k=2$, in equation (8.2.1), we have

$$\mathfrak{L}\left\{\left({}_1F_1\left[\begin{matrix}a\ \ ;\\ 2a\ ;\end{matrix}\ t\right]\right)^2:p\right\}=\frac{\left[\Gamma\left(a+\frac{1}{2}\right)\right]^2\ 2^{(2a-1)}}{\pi\ \ (p-1)}G^{3,1}_{3,3}\left(-(p-1)^2\ \middle|\ \begin{matrix}1,a+\frac{1}{2},2a\\ a,\frac{1}{2},1\end{matrix}\right)$$

$$=\frac{1}{(p-1)}\ {}_3F_2\left[\begin{matrix}a,\ \frac{1}{2},\ 1\ ;\\ 2a,a+\frac{1}{2};\end{matrix}\ \frac{1}{(p-1)^2}\right] \tag{8.5.26}$$

where $a+\frac{1}{2}\ \in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Re(p)>1$.

From case 1 to case 25 the condition of validity is $\Re(p)>0$.

I hope that the investigations carried over in the present thesis on multiple hypergeometric functions are of some interest and will provide insight into the structure of special functions in general and hypergeometric functions in particular and would motivate further research. It has been our aim to derive new results and to show that there is a fruitful interaction and connection between the hypergeometric functions of different types and phenomena nature. At the same time, we demonstrate that multiple hypergeometric series possesses a special character and has particular applications, which clearly makes it not merely a subtopic of curious generalization (in the study of hypergeometric functions), but a field in itself.

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Appendix

List of Research Papers Accepted / Published in the Journals

Certificate of Conferences / Workshops Attended